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HONG LIU:

OK, good. So let me first remind you what we did last time. So last time we discussed that, for any operator, say some operator O in the field theory, we expect this one to one correspondence to some field, say, ϕ in the gravity side.

And of course, the quantum numbers of those, they should match because of the four [? interpretations ?] of some symmetry groups and four [? interpretations ?] of the same symmetry groups. Of course, their quantum numbers should also match. And, in particular, on the field theory side, say if you have such an operator, it's natural to add the source term and then to deform the Lagrangian in such a form.

And then we explained that, this ϕ may be considered as the boundary value of ϕ in the boundary of ADS. So the boundary value of ϕ , and it's mapped to the-- if ϕ have some boundary value, then that is corresponding to your boundary series Lagrangian have such a term. You know, Lagrangian.

And then we also discussed this for the vector field, and for conserved current, say, the case which is a conserved current, then that implies that, in a particular example of this, if you have a conserved current that that's corresponding to, you must be dual to a gauge field. And in particular, the stress tensor must be to dual to the metric deformation in the field theory side.

So any questions regarding this? Good. OK.

So now let's try to develop more this relation. So this is normally called the operator field mapping. And so let's look at this relation a little bit more. So my immediate question, one can ask-- so the obvious question is regarding, say, the spin of the operator that should map to the spin of this operator, et cetera.

But if you can see the conformal theory, which is the scaling variant, a very important quantum number for scaling operator is the scaling dimension. As we mentioned before, we said, in the conformal theory, everything is scaling variant. And then you can assign any operator-- any operator can be transform under appropriate representation of the conformal symmetry of the

scaling dimension. That's one of the most important quantum numbers.

So now let's look at how the scaling dimension of this operator are related to the physical properties here. Because here, there is no low scaling dimension in the sense this is just some gravity field and here is isometric it's not the scaling symmetry. And then we also can try to understand the worries of properties of this ϕ , how they are reflected in O . So that's the purpose of today's lecture.

So for this purpose, now, let's look at the gravity side. So as we said last time, we always work in the regime of the semiclassical gravity. So on the gravity side-- OK. So what we will show is that, actually, the dimension of this operator can be directly related to the mass of the corresponding field in the gravity site. And so this is normally called a mass dimension relation.

So now let's just consider, on the gravity side, so the action can generically be written as the following. So everything is controlled by the gravitational interaction. So now I have introduced a new rotation, which we explain.

So you have Einstein gravity. And then you have some matter fields. OK? You have some matter fields. Matter Lagrangian. So this is the generic form, say, of action in the gravity side. So I've introduced this $2\kappa^2$, which is essentially whatever is the Newton constant in this d plus one dimension.

So normally we put the $16\pi G$ Newton constant here. So $d+1$ just means the Newton constant in $d+1$ dimension. And I just now called this $2\kappa^2$ for convenience.

And typically, the matter fields, say, suppose you have a scalar field, then you will have-- say for inform, plus, say, nonlinear terms. And if you have, say, if you have vector fields, then you will have a Maxwell actions, et cetera. So just whatever matter field you have in the gravity side, and they just have this-- so now let's consider small perturbations around the pure AdS.

So pure AdS, you only have a metric, and all the matter field is essentially zero. So now let's consider the small perturbations around the AdS. Or maybe I should say around the AdS.

So first thing you notice is that these $2\kappa^2$ is multiplied over all action. So that means that all the interactions here, essentially, are controlled by this gravitational interaction. Yeah, it's all controlled by κ .

And when you can see the small perturbations, say, when you can quantize the small

perturbations, et cetera, it's convenient to use a canonically normalized action. But if you look at the matter field here, the [INAUDIBLE] is this $1 \text{ over } 2 \text{ kappa squared}$, which is a lot convenient.

Actually, with that 2, then I don't need 2 here. So let me just get rid of the 2. So it's convenient to use a canonically normalized action. So it's considered a small [INAUDIBLE] around AdS, say, for convenient to canonically. normalize.

AUDIENCE: Excuse me, why--

HONG LIU: Yeah, one second. Normalize the kinetic term of small perturbations. OK? Which means it's convenient when we can see, say, for example the series for ϕ , when we consider scaling, they take ϕ -- say k , ϕ equal to $\text{kappa } \phi'$, and then you can see that in the ϕ' , we'll have a canonical action. And say, for example, if you can see the magic perturbations, and then you can write it as h_{mn} .

So this is the AdS matrix. OK? Then the ϕ' and h_{mn} will be canonically normalized. So later, we'll just drop this prime. I will just call this ϕ . So we just do a scaling operation. Yes?

AUDIENCE: Why did the 1 over 2?

HONG LIU: Oh no, it's just because there is 1 over 2 here.

AUDIENCE: But what path, you have r -- you say the r is in the action and r should--

HONG LIU: No, I'm talking about the matter field.

AUDIENCE: But in the action, you write the r minus 2 and the plus [INAUDIBLE].

HONG LIU: Yeah.

AUDIENCE: So ordinarily, should it be r plus 1 over 2 [INAUDIBLE] something else?

HONG LIU: It doesn't matter. It's just how I normalized things. You can normalize-- if you worry about the $1/2$, it doesn't matter. Here we're not worried about $1/2$.

AUDIENCE: OK.

HONG LIU: Good? So now here's the important point. So here is the important point. And remember, when we have the relation between the Newton constants, so the Kappa squared is a Newton

constant. And remember, this is proportional to our n squared. Remember what we discussed last time.

So that means, under this scaling, so κ is actually of all the 1 over n , κ is the Newton constant it's a square of Newton constant. It's very small. So this is in units of literature. And we always can see that in the large N image so that the Newton constant is small.

And so this is small. So in this regime, if we keep ϕ -- so for canonically normalized field, say, if [INAUDIBLE] ϕ are of order 1 , then their perturbation is naturally small. Then their perturbation is naturally small because they are essentially controlled by the κ .

And it also means, if you plug this into the action, so it expands also long linear terms. So it also means that nonlinear terms are order κ or higher. So this is very easy to understand.

So let's consider, say, ϕ cubed term come from here. So previously, we have 1 over κ squared, ϕ cubed term, so after this scaling, then I just have $\kappa \phi$ cubed. So for ϕ of all the 1 , then the cubic term will be suppressed compared to the quadratic term because the quadratic term, after the scaling, will be of all the 1 , will be all the 1 . So the λ near term will be suppressed or higher and suppressed.

So except if we are interested in considering the contribution of those higher order terms, then to leading order, we can actually just consider, for small fluctuations, we can just consider the quadratic action. This means that to leading order?

So we can consider the quadratic action. So this means we can just-- so this corresponding to quantize the free series, OK? So this corresponding to quantize the free series.

Remember, we always use \hbar equal to 1 . So for those matter fields, \hbar is always equal to one. And after you do the scaling to get rid of the κ , we just treat them as the free field theory quantization. Good? Any questions about this?

So now let's look at scalar fields, just for illustration. Let's look at what's the behavior of a massive scalar field. So now let's consider a massive scalar field. for illustration.

And we consider some scalar field ϕ in the [? back, ?] which is due to some scalar operator O in the boundary. Just imagine. OK? So now let's consider the quadratic action for ϕ .

So we can write down the quadratic action. So after you have scaled out this κ , and the

quadratic action can be written explicitly as the following across all the kappa terms.

So this is just a free action. And of the GMA, just the AdS. So the quadratic order, this just in the background AdS. So we'll use this notation to denote the [INAUDIBLE] component of AdS.

So we're most of the time using this metric. This is my background AdS metric. And this is my action.

So we'll also use of the notation that the x^μ actually, I should write this matrix here, but it doesn't matter. The x^μ should be considered as z^μ . And x^μ is the coordinates parallel to the boundary.

So do you remember how to quantize the free field series action? So this is the time we need to remember that. So the first thing is how you solve the equation motion.

So we actually don't need to go to details. I will just mention a few important features. OK, so this is just the standard [INAUDIBLE] equation for massive scalar fields in the curved spacetime. And this gMN is just given by this metric.

So this is a seemingly complicated partial differential equation. But here we have lots of symmetry. So there is a translational symmetry along the directions parallel to the boundary. Then we can just write down a Fourier transform.

So because of the translation symmetries in x^μ directions. So we can just do a Fourier transform. We can write $\phi(z^\mu)$ in terms of, say, $\tilde{\phi}(k)$.

So this $k \cdot x$ is just the standard Minkowski contraction because of the t and x part. And essentially it's slightly a Minkowski metric. So [INAUDIBLE] here, $k \cdot x$ is just the standard Minkowski contraction.

So now it's the first transform. Then you just plug this in. Then this is very easy to write down. The only derivative now would be just in the z direction. Because then all the derivatives in the t and x can be replaced by k .

And then the equation becomes very simple. So it takes you a couple of seconds to write it down. So let me just write it down. So it can be written in the following form.

And then the k^2 just is the standard one. Again, the k^2 is just as the standard Lorentz contraction. So ω would be the k^0 in the time direction. We're just do the

standard Lorentz contraction. So $k \mu$.

So now, this equation can actually be solved exactly. So normally, when you quantize it, you solve the equation motion, then you find the complete basis of solutions. Then you expand in terms of that complete the basis, the co-option of the expansion would be your creation and annihilation operators. And then that's how you quantize this series.

So this equation can actually be solved exactly. So you can actually easily find out the complete set of modes. But it would be actually not needed for the current purpose right now. So maybe not solve it. So we'll leave it here.

But what will be important for us is consider the asymptotic behavior of the ϕ , the ϕ near the boundary. Now, let's consider the behavior of a ϕ near the boundary, which will be very important for us. Now let's just consider behavior of ϕ near the boundary.

So near the boundary, z goes to 0. So this term compared to this term is always much smaller because z goes to 0. $m^2 R^2$ is finite. So we can drop this term. But kinetic term, typically, cannot drop because kinetic term depends on $1/z$.

So we just throw away this term. So as z goes to 0, and maybe also expand this term more explicitly. Then what you find, just drop this term and then write in this term more explicitly. Then you find the equation can be written as following.

So this is only valid when z equals to 0, leading order. So now this is equation has a very nice form because this equation is homogeneous. So z^2 , then the partial z^2 , z partial z , and then you have this thing which there's no derivative. And so this is not in some sense, homogeneous in the derivative. Such kind of equation you can always solve it by some power of z .

So this just write down answers. So assuming ϕ is proportional z to some power. And this power, let me call it δ . So you plug these answers into this equation, then you get [INAUDIBLE] equation. So you can easily see, from first term, you

get δ , $\delta - 1$. From the second equation, you get $1 - \delta$ times δ , $m^2 r^2$. And this is a simple quadratic equation now you can immediately solve. So you found two solutions. Find two solutions.

So we are writing this in the form $1/2 \delta \pm \nu$. I will define this thing to be my ν , so

this is definition. So I will introduce the following notation. I will take the plus sign. I call the plus sign delta.

So from now on, delta refers to plus sign. And sometimes I use the notation called delta minus, which is the minus sign, which is also equal to d minus delta. So this is the definition. So now I use this as the definition. So from now on, my delta is always d divided by 2 plus this number.

So now what we have found is that as z goes to 0, what we have found is that as z goes to 0, that destabilizes. So as z goes to 0, the ϕ has the following behavior. So you have two independent solutions. So then I can put arbitrary [INAUDIBLE], $z^{d-\delta}$ to exponents, and another one is $B_k z^\delta$. Then plus have all the corrections. OK?

So this is two leading behaviors in z goes to the zero limit. And of course, I can also go to the coordinate space. Then this is just the x, z , which is $A_x x z^{d-\delta}$ OK?

So first notice that this exponent, this $d-\delta$ exponent is always smaller than delta. So this term always dominates. So this is the leading behavior. And this is the sub-leading behavior. So keep this in mind.

So now we have found the fourth asymptotic behavior of the solutions. And now we can discuss physics. Now we can discuss some physics.

Now let me make remarks. For the first remark-- any questions so far? Good. So the first remark, you notice that this exponent are real if $m^2 R^2 > d-4$.

Then you say, ah, for sure, these exponents are all real. Because normally, we only consider the positive mass squared. But [INAUDIBLE] idea is something very interesting happening. It's actually, in AdS, the [INAUDIBLE] mass squared also makes sense.

In fact, you can show, as far as those exponents are real, means that, as far as, so one can show, as far as the star is satisfied, the theory is actually well defined.

So in AdS, you are now-- you have an [INAUDIBLE] mass squared. But if you violated this, then you find instability. This is a very nice story, but I don't have time to discuss in detail here.

But my former student [? Nabil Ackbar ?], when he was doing TA for me last time, he wrote a very nice note explaining how this works. And I put that note on the web. So this equation is called the BF bound. F is of our colleague Freedman. And the B is a very complicated name I

always pronounced wrong. So maybe not try here.

Anyway, so [? Nabil ?] gave a very nice explanation of where this comes from. And he shows that when you violate this, you actually get instability. And this maps into a very nice quantum mechanics problem, the quantum mechanics problem of minus 1 over x squared. Some of you may have experience with that problem.

Anyway, so please take a look at those. I put it on the web. And it's a very instructive discussion. But we'll take a lecture to do that.

Anyway, so now I just make the claim. It's that, as far as those exponents are real, the theory's well defined. OK? But if this exponent is violated, then there exist modes which are exponentially growing in time indicating the system is unstable. So this star is called the BF bound. The BF bound.

So it's actually using instructive to compare with a standard-- actually, I forgot to use this blackboard. Anyway. Yeah. So [INAUDIBLE] to compare with Minkowski spacetime, so this compare with Minkowski case.

So, in the Minkowski case, if you have a free particle, say, if you have a free massive scalar field-- so of course, we know this is our equation of motion, OK? And from here, if you go to the Minkowski space, every direction is translation invariant, then you just get this dispersion relation. You just get this dispersion relation. You can write $\phi = e^{i(\omega t + \mathbf{k} \cdot \mathbf{x})}$. Let me just write it more carefully here. You can write $\phi = e^{i(\omega t + \mathbf{k} \cdot \mathbf{x})}$. Then plug it in. Then you find for $\omega^2 = m^2 + k^2$.

So in Minkowski spacetime we don't allow m^2 to be smaller than 0 because if m^2 is smaller than 0, then that means, for certain range of k , say for k , for those values of k which are smaller than the absolute value of m^2 , then this will be negative. So for those things, the ω will be purely imaginary. You can be plus minus purely imaginary.

And then you plug it in here. Then you find it in ϕ and typically have exponential $e^{\omega t}$ plus exponential minus $e^{-\omega t}$. So the ϕ can have these two behaviors. And then, of course, this will exponentially grow. So this tells you that the magnitude of ϕ will exponentially grow with time. And so that indicated this instability.

So this is the same as the analog I said earlier before. In the Minkowski spacetime, if you have a mass squared, and then your scalar field one wants to slide down, and this just tells you the

behavior wants to increase.

So this the ϕ . So this is $V \phi$. Then the ϕ wants to increase. So this is ϕ to 0.

But in AdS, something nicer happens. So in AdS, the instability only sets in if the mass squared becomes sufficiently [INAUDIBLE]. So if you just have the [INAUDIBLE] mass squared, if it's greater than this value, it's still OK. It's still OK.

So the reason is actually very simple. So the reason is very simple. You can try to convince yourself by looking at the equations. And you can see the physics from the equations. But the basic physical reason is following. In AdS, so you can just look at that equation. Because of the spacetime curvature-- so the constant modes are not allowed. So you see here, the most dangerous modes are the constant modes. And when you have constant modes, and then m squared becomes-- no matter how small it is, no matter how small m squared is, if you have constant modes, and constant mode means k equal to 0, then that means that ω squared will always be negative.

As far as m squared is nonzero in the negative, there's always constant mode which is unstable. But in AdS, because of the spacetime curvature, the constant mode is not allowed. So you always have some kind of kinetic term in the sense that you always have some kind of kinetic energy.

So a field is forced to have some kinetic energy, which can compensate this negative m squared. But this won't happen when the mass becomes too negative. And then you have instability. Then you have instability.

I suggest to you-- so this is just words, but those words can be reflected by looking at the structure of those equations and try to get some feeling about these by looking at those equations a little bit more carefully. OK, so this is the first remark. This is the first remark.

So the second remark, is that we know that the AdS have a boundary. And the light array which of the boundaries in the front of time. So this is the one important feature we discussed at the beginning, which is the boundary in the finite time. In fact, in the AdS, it's just π over 2 in the finite time.

So that means, actually, energy can be exchanged on the boundary. Essentially, you can send stuff in or send things out. And so we actually need to impose the-- so in order to have a

sensible physical system, you need to impose appropriate boundary conditions between appropriate boundary conditions.

So the most important basic criterion-- again we are not going to the detail. And again, [? Nabil's ?] notes discussed this because he has to-- in order to discuss instability, he has also to discuss the boundary condition, et cetera. And you can look at [? Nabil's ?] notes.

So basic criterion to impose a right boundary condition is to make sure that your energy is conserved and that your energy don't dissipate into nothing, don't dissipate at the boundary so that you have a well-defined system. Good.

So now. So when you do canonical quantization, then what do you do is that you expand ϕ in terms of a complete set of normalizable modes. Of course they also have to be normalizable as, normally, in the quantum mechanics, normalizable modes which satisfies the appropriate boundary condition.

So before talking about the explicit boundary condition, let me first mention that issue of normalizability. So to talk about normalizability, you have to devalue in the product. So here we have the standard inner product for quantum field series in the curved spacetime. So here we have the standard inner product. And let me just write it down.

So this is not specific to AdS. This just works for any quantum field theory, for any free scalar field series in the curved spacetime. So, for that series, you can define an inner product that follows. So consider. You have two modes, ϕ_1 , ϕ_2 . And then the inner product can be defined as the following. You choose a constant the time slice. You have to choose a constant time slice, which I called σ_t . So this is a constant slice.

By time, I really mean this time. You choose a constant time slice. And then, of course, you integrate over spatial coordinates on these constant time slice. So this is the inner product. So this is the standard inner product. And you just plug in the AdS metric that gives you the explicit form for the AdS.

So if you're not familiar with this, and you can check that this makes sense, that by showing that, actually, this thing is actually independent of t , so it does not depend, actually, your time slice. Because you don't want to your definition of inner product depend on specific time you choose.

Then the normalizable would be the standard condition, whether this thing is finite, or infinity,

et cetera. And you plug-in some modes with itself. And if it is finite, then it's normalizable. And if it's infinite, then it's not normalizable.

So now this is our asymptotic behavior. And you can check yourself, you can also convince yourself that the most dangerous behavior because the metric diverges near the boundary. So the most dangerous part of this integral is near the boundary because your metric diverges there. And you can just plug this behavior and this metric behavior. And then just pay attention to the behavior near the boundary. Then you can decide which mode is normalizable, which mode is non-normalizable.

Then you find, again, this takes you a few minutes to do. So you'll not do it here. So I will only talk about the real ν . From now on. Just real ν . It means that we always satisfy that bound. So we find that, for that mode, so this mode always goes to 0.

So you have these two modes. This mode always goes to 0 because δ is always positive because ν is positive, and ν is non-active, and $d - t$ over 2 is positive. So this is always goes to 0. And you can show that this is always normalizable.

OK. So there's no divergence at the boundary. But for these modes, you're going to see these modes become dangerous when the mass become very large. When the mass becomes very large, then the ν is big. When ν is big, eventually the δ can be pretty big. And then can be greater than d . And then this will be an active exponent. And then this will blow up. So it turns out this is actually non-normalizable indeed.

So this is non-normalizable for ν to be greater than 1. You actually don't need to for, mass to be very large, because for ν to be greater than or equal to 1, this is already non-normalizable. But it's an interesting thing. You said this mode is also normalizable when ν is in this range.

Good. Any questions? Yes

AUDIENCE: Can we define inner product some other way?

HONG LIU: No, this is the only way. This is the canonical way, just generic for quantum field theory in the curved spacetime. This is not specific AdS. Yeah, essentially, this is a unique one, yeah I would say.

AUDIENCE: So I'm just trying to understand inner product. In the case that I take, make it a norm in the

sense that I take inner product of a field by itself. So why, for example, should this be positive?

HONG LIU: No, no this is in general not positive. In general, this is a what we know in the Klein-Gordon field, say, if you do in the Minkowski spacetime, this in general, not positive definite.

AUDIENCE: Right, sure.

HONG LIU: Yeah, same thing in the Klein-Gordon fields. Yeah, but the [INAUDIBLE] is, so that means this thing does not make sense to interpolate as a probability. But you can still use this as a definition of normalizability.

AUDIENCE: So it still does the property that something will only have so it's not a semi-norm in the sense-- like it is actually a good inner product?

HONG LIU: Yeah.

AUDIENCE: Isn't it this it just the generalization of probability current of [INAUDIBLE]?

HONG LIU: Yeah, no, this is the same as the Klein-Gordon equation. This is the same inner product as you would define for the Klein-Gordon field, a free massive scalar field in Minkowski spacetime. And that this is just the generalization to a general curved spacetime. Good.

So now, yeah, so we see that this is always normalizable. And this sometimes is normalizable and sometimes not normalizable. So now, let me talk about the boundary conditions. So for the time reason, I will not show that the boundary condition I'm going to talk about will give you a well-defined energy in the sense that there's no energy exchange at the boundary. And for that, take a look at [? Nabil's ?] notes. I will not go into detail on that. But the boundary condition I say will satisfy that property.

So let's first look at when ν is greater than or equal to n . In this case, the answer is obvious. You don't even need to check the energy conservation. Because you only have one possibility because this mode is non-normalizable. And a non-normalizable mode, you cannot keep them when you do the quantization. Quantization, you have to use normalizable modes. So the boundary condition you have to choose in order that conical quantization is equal to 0.

So you should only keep this mode of a ν equal to 1. Because this is non-normalizable. So now, when ν is in this range, so both modes are normalizable, but you cannot choose both of them. You cannot allow both of them because then it leads to the non-conservation. You need

to put some boundary conditions there.

So then, naturally, the equal to 0 also works in this range. So this is called the standard quantization. You can still impose, even though the A modes become normalizable, you can still impose $A = 0$. So tentatively you can also impose $B = 0$. Just take one of them. Because A now is normalizable. So this is called the alternative quantization. So sometimes, it's also possible to impose the mixed boundary condition between them-- to impose a mixed boundary condition. I will not go into there.

Anyway, so for example, if you look at [? Nabil's ?] notes, you can easily check for those boundary conditions that there's no energy flux, through the boundary. So, from now on, I will use the following terminology. From now on, we'll use the following terminology.

So from now on, when I say normalizable, I always say the mode which are picked by quantization, OK? So when I say normalizable, in this case, of course, this is normalizable. When I say normalizable, for ν greater than 1, it's no ambiguity, just the B mode. And when ν is between 0 and 1, the normalizable refers to the mode which you choose to quantize. And then the other you consider to be non-normalizable. Is it clear, the terminology?

So in the standard quantization, the B-mode is normalizable, and the A is non-normalizable, even though, mathematically, A is normalizable. And in the alternative quantization, the B is non-normalizable, even though, mathematically, it is normalizable. So normalizable would mean, the behavior specified. I'll call the other modes non-normalizable modes. So with this terminology, I always have 1 normalizable mode, one non-normalizable modes.

OK, let's move on a little bit more. So this is the second remark.

And the third remark-- let me just erase this. So the third remark, so normalizable modes, by definition, is the one we use to do a canonical quantization. And it's the one we use to build up of the Hilbert space.

But now, since we have an equivalence between the two series, the Hilbert space of the two series will be the same because, otherwise, the state of the two series should be the same. Otherwise, we cannot talk about the equivalence. So that means that any normalizable mode, so anything which fall off, say, as, for example, in the standard quantization, like z to the powers δ at infinity near the boundary, you should consider as mapped to some states of the boundary theory.

So this is, actually, a very general statement. So this will also work if you include those higher order corrections which are suppressed by κ . So, in general, you can apply it to a complicated interacting nonlinear series in the gravity side. And the normalizable modes are always the modes which are in your Hilbert space in the gravity side. And so that means the mass corresponding to some states in the boundary field series. Good.

So now, let's look at the interpretation of the non-normalizable modes. Non-normalizable modes, suddenly they are not part of the Hilbert space.

So if they are present, in particular-- let me just finish this. then we'll say it. So, if present, it should be considered-- so sometimes, they're just not present. But, if in some situations, they are present, then you should view them, view such kind of non-normalizable modes-- if I'm in the background.

So by background, for example, the AdS, pure AdS, which we are expanding around is our background. Yeah. So you can imagine, say, [INAUDIBLE] non-normalizable modes deform the AdS, then you have a new background, then you have a new background.

And in that background, you're not supposed to quantize the cell, because these are non-normalizable modes. These are non-normalizable modes.

AUDIENCE: Yes, sorry, I guess I didn't quite get the terminology normalizable for 2, I guess? You mentioned it, but could you repeat it?

HONG LIU: Sorry?

AUDIENCE: Normalizable, not normalizable.

HONG LIU: Yeah. Normalizable means the behavior specified by the quantization.

AUDIENCE: So I don't think-- I don't quite understand--

HONG LIU: Yeah, let me just repeat. For ν greater than 1, normalizable, it's just normalizable, because there's only one normalizable mode. For ν between 0 and 1, mathematically, both modes become normalizable. But when we quantize them, and we [INAUDIBLE] one of them. OK?

As I said, in order to preserve the energy conservation-- so we have to select one of them. So there are two possibilities. You select the standard quantization, which you said A equals to 0, you said, the alternative quantization you set B equal to 0. So the 1 you set 0 is considered to

be non-normalizable in this terminology. And the 1 you allowed, which means the one you are used to expand as a canonical connotation are called normalizable.

I will elaborate this a little bit more. Yeah. Yeah, after five minutes, then you will see. Yes?

AUDIENCE: What do you mean by the other ones being present? If we don't quantize them, have they appeared?

HONG LIU: Yeah, they can appear. I will, again, explain in one minute. So now, I'm going to make a connection for this statement to the statement we are making last time. We're making last time. So last time we said, if a gravity field has a boundary value, then this boundary value can be considered as a source multiply the operator which you can add to a Lagrangian. You can add to the action.

And so this is precisely like that. It's precisely like that. So let me just repeat. Say, let's consider the standard quantization just to be specific and say standard quantization. In this case, the non-normalizable mode corresponding to $A \neq 0$, so if the non-normalizable modes are present, and this is corresponding to $A \neq 0$.

And we see from here, this term, A term, always dominated the other term. So, in other words, when you go to the boundary limit, it's the A which determines the boundary value of the field. Let me just say. The A , which is determined boundary value of the field. I put the code.

AUDIENCE: [INAUDIBLE].

HONG LIU: $A = 0$ when you do the canonical quantization.

AUDIENCE: Oh, OK. [INAUDIBLE]

HONG LIU: But $A = 0$, you mean this is the condition in the normalizable modes. So here, we are discussing the interpretation of the non-normalizable mode. So, now, suppose A is nonzero. Suppose a non-normalizable mode is non-zero. Suppose, now, the non-normalizable mode is non-zero. We are down with normalizable modes. This is a condition for the normalizable modes. OK?

So since this term dominates, the A should be interpolated as a boundary value of the fields. And if you have a non-zero A , so if you have a non-zero A , say, if $A x$ is equal to some ϕx , according to what we discussed last time, now up to this kinematic factor, this thing, which you

need to strip away because this just determines your equation motion.

So if A equal to 5 equal to 0 , that implies that the boundary series action should contain the term. So what this means is that the non-normalizable, there are certainly not part of the Hilbert space. If they are present, they actually determine the boundary theory itself.

So the non-normalizable modes determine the boundary theory itself. So we will later see examples of this. But let me just repeat, so if you look at the gravity solution, then you look at its asymptotic behavior near the boundary.

And if you see-- let's, say, consider two solutions, two different solutions. If the two different solutions have the same non-normalizable behavior-- say the non-normalizable modes are the same, but only the difference in the normalizable modes, then you will say these two solutions describe two states of the same theory. Because they only differ by normalizable modes.

Then it describes the different states in the same theory. But now, if I look at two such solutions, and these two solutions differ by the behavior of the non-normalizable modes, then we would say, these two solutions corresponding to two different series because one is due to two different series. Because non-normalizable modes needs to add additional terms to your boundary theory action. Have I answered your questions?

AUDIENCE: So basically, you just assume any field which is a type of-- maybe a combination of those two modes, but if the--

HONG LIU: Forget about linear combination of the modes. Just say, if you consider two solutions, which if there only differ by normalizable modes, then they correspond to different states of the same theory. And if they differ by non-normalizable modes, that means they describe different theories. Even the theories they describe are different.

And so that's why we call this background. We call this background because the asymptotic conditions at infinity has changed.

AUDIENCE: I have a question. So, in the alternative standard upon the quantization case that-- which would be equal to 0 .

HONG LIU: Right.

AUDIENCE: But we know that the d minus δ is always larger than δ . But in that case, it seems near

the boundary. The boundary determines, not the number.

HONG LIU: Right, right. Yeah. That's a very good question. So that, you have to stretch a little bit. Yeah. Yeah. Yeah. Good? So let me finish my last equation before we break. So more precisely, because of this z term, so in the standard quantization, yeah let me just write it there, so write it here. Let me just replace this one.

In the standard quantization, if you have such a term in your boundary series Lagrangian, that's corresponding to this ϕ is equal to $\lim_{z \rightarrow 0} z^{\Delta - d} \phi$. So I have to multiply a power so that it will just extract this A factor from the behavior of the ϕ .

And the so this is the relation. So this is the important relation. So, last time, we only said that ϕ should corresponding to the boundary value of this capital ϕ . But now, by working out the asymptotic behavior, we have refined that statement. Now it's a really precise mathematical statement. Any questions regarding this?

AUDIENCE: Is it a guess or--

HONG LIU: No, this is just a [INAUDIBLE]. This limit precisely extracts this A term. This term, because that opposite power to this one, and this one will be smaller. This one will be smaller because that multiplier, this one, when z goes to 0 will go to 0. So you automatically extract this term.

AUDIENCE: [INAUDIBLE]

HONG LIU: It's just this statement. The A is related to this. OK.

AUDIENCE: No, this statement is a guess from the example we discussed last time. So last time, by looking at this line example, we deduce that the boundary value of a field should be related to the source. The boundary value of a [INAUDIBLE] field should be related to the source of a boundary operator.

And at that time, we don't know what we mean, precisely, by boundary value. And now we have worked out a precise asymptotic behavior. And now you can talk about what you mean by boundary behavior. And then that refines that statement to that precise mathematical statement.

Good. So let's start. So let's see what we can deduce from this relation. So turns out that

there's a very important information you can deduce from this equation. And this is my number five.

So let me call this equation star. So this equation star, relation star, actually tells you that this delta, which defined before, in this way, and appears in your asymptotic formula like this, this delta is actually, it's the scaling dimension of the operator dual to ϕ of \mathcal{O} . So we're assuming this ϕ is due to some operator \mathcal{O} in the boundary. And so this is actually this precise in the scaling dimension of \mathcal{O} . So now, let me explain that.

So before I do this, do you have any questions? Good. OK, so let me first remind you how we define the scaling dimension of an operator in conformal field theory.

So if you have a scaling [INAUDIBLE] theory, then the overall scaling is a symmetry. So such a scaling is a symmetry. And so under such symmetry transformation, say, operator, we call it a scaling operator if you transform. And under this symmetry has the following. So transform into a new operator prime. And evaluated at the new x prime is equal to just differ by a scaling from the previous operator.

OK, so we pause. So the operator, with such property, a scaling operator. So good operators, which are representations of the conformal symmetry always have this behavior. And this delta defines the scaling dimension.

So this delta is the scaling dimension. So this is the definition of the scaling dimension. So this is almost similar to a scalar field. So a scalar field means that the new scalar field, you evaluate it at a new point. It should be the same as your old scalar field. You evaluate it at the old point.

And here the difference is just of a pre-factor. And this number defines the scalar information. Good? So now I will derive that this relation implies that this operator transforms precisely according to this delta.

So let me first I remind you that the boundary scaling x^μ goes to x^μ prime equal to λx^μ is related to the bark, also to isometry, which is the x^μ prime equal to λx^μ . But, at the same time, you also scale z .

So this scaling symmetry, the counterpart of this scaling in the gravity side is that you scale both of them together. And the gravity metric is invariant under this. So now let's consider transformation. So now let's consider the relation between the $\phi(x, z)$ and the $\mathcal{O}(x)$.

So let's consider such a transformation on both sides. So on the gravity side, such a transformation leads to a new field. And this should be dual to transform the field on the field theory side.

AUDIENCE: Which [INAUDIBLE]

HONG LIU: Yeah, that's right. Yeah. Yeah. That's right. Now, this is [INAUDIBLE]. This is a boundary. OK?

But I think, since this is a symmetry, so this transformation of both symmetries in the field series side on the gravity side, that should have low consequences. That should have low consequence. That means. That means.

And the ϕ' , this $\mu\phi$ is the boundary value of the corresponding capital ϕ , OK? This should still be the same because your physics should not be changed under this. So now let me remind you that ϕ is a scalar field. In the gravity side, this is a scalar fields. So scalar fields transform, under such a coordinate transformation, transforms as ϕ' , X' , z' . You go to $\phi \times z$.

So this is just the definition of the scalar field in the gravity site. And now let's apply this to the boundary value. Then we find that the $\phi' x$, which is a boundary value capital ϕ' , so this should be equal to that.

So now we use that this relation of z' is λZ . And this is just equal to that. So this just becomes λ to the delta minus $t \phi x$. Clear?

So now you can just plug this back into here. So there's a factor of λ minus delta minus d . And this comes with a factor of λ to the power d and then tells you that $O' x$ prime equal to λ minus delta $O x$.

And we show that this delta, which appears here, or, whatever, here, is that delta. It's the dimension of the operator.

AUDIENCE: But in standard quantization, you mentioned that--

HONG LIU: No, this is a standard quantization.

AUDIENCE: A equals 0 in standard quantization.

HONG LIU: No, no, we're talking about the source. We're talking about non-normalizable modes. We're not talking about canonical quantization. We're not talking about canonical quantization. We're talking about attending non-normalizable modes, because when we're adding on such a term to action.

AUDIENCE: But isn't that a field operator correspondence?

HONG LIU: Sorry?

AUDIENCE: Isn't that a field operator correspondence?

HONG LIU: Yeah, this is.

AUDIENCE: But these are non-normalizable.

HONG LIU: Yeah, that's what we, yeah, yeah, yeah. Yeah. So we're talking about the non-normalizable modes. And the non-normalizable modes corresponding to [INAUDIBLE] operator, and [INAUDIBLE] source into an operator in the boundary series. And just this relation shows that delta is this dimension. Any other questions?

AUDIENCE: So then, when we quantize everything, this operator O is dual to normalizable [INAUDIBLE], or non-normalizable?

HONG LIU: No. Here we are talking about the question-- here we're talking about this relation that if there is a boundary value of capital phi, which is this boundary value, then that's corresponding to add such a term to your Lagrangian. And under such a correspondence, then we deduce that this delta is the dimension of this operators. It's the dimension of this operator. And this is deduced from the non-normalizable perspective. And in indeed, you can ask, say, suppose I create normalizable modes using this O, whether that is consistent with that. Indeed, that you can double check. Is it clear?

Anyway, so this is what we have right now. So later, we will discuss the correspondence for the normalizable modes. And we worry about later. Later, we will talk about the correspondence for normalizable modes. And then you will see it's compatible with this relation.

So now we find, for scalar, for a standard quantization, for scalar, in the standard quantization, for scalar in the standard quantization, that the dimension of the operator related to the mass in the gravity side by the following relation.

[INAUDIBLE] to [INAUDIBLE] by this relation. So we have found that explicit relation between the conformal dimension and the mass. So now, let me elaborate a little bit on this relation.

So let's consider, suppose the m equal to 0. Well, m equal to 0, then you find the Δ equal to d . So Δ equal to d , on the field theory side, this is a marginal operator, and so we see that the massless field on the gravity side is mapped to a marginal operator. So are you familiar with the terminology of marginal, operator, relevant operator? Et cetera. Good.

AUDIENCE: Wait, can you remind us what is it?

HONG LIU: This is defined in terms of the RG flow. Are you taking the quantum field theory too?

AUDIENCE: Yes. We haven't quite gotten there. [INAUDIBLE].

HONG LIU: Yeah, then you will get there, maybe, in a week or two. Yeah, certainly by the end of this semester. Yeah, so let me just say one-- yeah I will say a few more words. And then you will see what it means. Yeah, just I will say a few more words.

Anyway, so will m^2 smaller than 0, then Δ is negative. Then this is d^2 over 4 matter something. So that means the Δ is smaller than d .

When the dimension of the operator is smaller than the spacetime dimension, so that, from field theory, according to you, is called the relevant operator. And now, when m^2 is greater than 0, it's mapped to Δ greater than d , so this is mapped to an irrelevant operator.

AUDIENCE: Why do they have [INAUDIBLE]?

HONG LIU: Why do they have these names? Marginal, relevant, or irrelevant? Yeah, I will say a little bit about that. Just wait one second.

So now, let's say what this means from the perspective of this relation. So from the field series side, we have $\phi \times O$ mapped into-- so this ϕ should be identified with A . So this is identified with this term, $d - \Delta$. OK, leading term.

OK, so now let me remind you what this marginal and relevant, irrelevant means. So operator marginal means, when you add such a term to your Lagrangian, the effect of such a term does not change with scale. So whether you're looking at a very high energy, looking at very low energy, the effect does not change very much.

OK, so that's why it's called a marginal operator. So in the regulation group, there's a very simple scaling argument to show that, when the delta is equal to d, and then the effect of the operator does not change with the scale.

In particular, when you go to the UV, this does not change. For a marginal operator. When you go UV, say, when you go to higher and higher energies, the effect of these operators remains the same, roughly.

So now, we know from this UV connection, UV mapped on the gravity side to z goes to 0 means we are approaching the boundary. And now we see, precisely for d equal to Δ , this term is actually constant. So go to smaller and smaller z , and the effect of this term does not change. It's just a constant. So again, does not change.

So now, let's look at the relevant operator. So the relevant operator, so in the field theory, when you do the RG, the relevant and the irrelevant means, whether this term becomes relevant when you go to the IR. By relevant, means that the term becomes more and more important when you go to the low energies.

So that means that, for the relevant operator, when you go to higher energies, it becomes less and less important. By definition, the relevant operator means, when you go to low energies, it becomes more and more important. And it means, when you go to higher energies, it becomes less and less important. So it means that, when you go to UV, the relevant operator becomes less and less important.

So now, let's look at this side. The relevant operator corresponding to Δ smaller than d , then you have a positive power. And that means, when z goes to 0, when you go to UV, actually, this term goes to 0. So this is precisely compatible with the field theory behavior we expect when you add such a term to a Lagrangian. And the effect of such a term should be less and less important when you go to higher energies. And here, we see a precise counterpart here.

And similarly, now, if you go through the irrelevant operator. Irrelevant means irrelevant when you go to IR. When you go to UV means more and more important. So that means-- but here, when Δ is greater than d , then these here have negative power. So here, goes to infinity, become bigger and bigger and, indeed, this term becomes more and more important. So we see a very nice correspondence between this behavior we expect just from the ordinary RG and the behavior of this term in the gravity side.

So this is a very important self consistency check of our matching, both of this relation and of this relation. Any questions on this? Yes.

AUDIENCE: What's the topology of AdS boundary?

HONG LIU: Sorry?

AUDIENCE: What's the topology of AdS boundary?

HONG LIU: It's a Minkowski space.

AUDIENCE: But it's 5-- it's 5 times something.

HONG LIU: It is a Minkowski space. Boundary of AdS is just Minkowski space.

AUDIENCE: I have a question. [INAUDIBLE] in RG why [INAUDIBLE]

HONG LIU: That take some story to do. Yeah, that you have to do in the quantum field theory too. Yeah, but it's very easy to understand if you just know a little bit of free field theory. And in the field theory point of view, when the dimension of this operator is smaller than the dimension of this operator, and then this is dimensional, then this has a positive mass dimension. And this is, like, mass term.

And the mass term becomes more and more important when you go to lower energies. And that's important because the higher energies. Yeah, just from dimensional analysis. Yeah, it takes maybe five minutes to explain very clearly. But we don't have these five minutes.

OK, so let me just summarize. So we have this relation. Let me maybe just summarize here. So we have the other relation. So summary. So we have ϕ due to go to some O . So here, normalizable modes should be considered different normalizable modes corresponding to different states.

And here, non-normalizable modes is mapped to different actions. So you have a different non-normalizable modes, then you map to different actions. You map to different series. So in other words, map to different series.

So when you turn on the non-normalizable modes, you are deforming the theory. So just to elaborate, just more precisely, in the standard quantization, which is A modes are non-normalizable, when the A -modes are non-normalizable, then $A \times$. On the gravity side, is

mapped to this last term on the field theory side.

If you have a non-0 x , $A x$ mapped to the following term in your action. And then your mass, we deduce that this side is mapped to the dimension, give the delta dimension of this operator O .

So later we will explain, as part of this normalizable mode story, so this B -- so A is non-normalizable but B is normalizable. So later, as we will explain, in the normalizable mode story, is the $B x$ here, we are actually mapped to the expectation value of O .

So B is a normalizable mode, then, actually, that maps to a expectation value of O in the corresponding states. So A mapped to how you deform the theory, and the B just mapped to the state which specify your expectation value. So this, we'll see later.

And in alternative quantization, you just do it oppositely. In alternative quantization, you just switch A and B . In alternative quantization, you just switch A and B . So now B maps to $B x O$. And $A x$ maps to the expectation value. And m now maps to d minus delta.

So now, the dimension of the operator is d minus delta rather than delta. So now, because you just take the different exponents. It's the same thing, just take different exponents. So now, the dimension of the operator is d minus delta.

So now, let me also make you some other examples. So you can consider gauge fields. You can consider, say, a Maxwell field in the gravity side. Then that's [INAUDIBLE] dual to some conserved currents, so besides earlier. And then you can check, then again, you just solve the Maxwell equation, which I will not have time to do it here.

But I may ask you to do it in your p set. It said $A \mu$ as z goes to infinity becomes-- so the path of A parallel to the boundary goes to a constant. There's no z . It's really a constant plus $b \mu$ some number to the z to the d minus 2. So this is a leading behavior in the case of the vector field.

So using the very similar scaling argument I was doing there, you can deduce that the operator due to A must have dimension d minus 1. This I will leave to you as an exercise. Just exactly parallel statement for the scalar case. And with this asymptotic behavior, you can show, just using the argument, you can show that the operator dual to A must have dimension d minus 1.

And this is precisely the dimension a conserved current must have. So conserved current because it's conserved. So its dimension, it's actually protected. So no matter in what kind of interaction, no matter in what kind of case, you always have the same dimension $d - 1$. So this precisely agrees with that. So do you understand why?

AUDIENCE: Why.

HONG LIU: Yeah, the reason is simple. It's just because j_0 , the dimension of the j_0 . You can understand the following. The j_0 is the charge density. And the charge density charge, by definition, does not have a dimension. And by definition the density will always have a dimension minus 1, $d - 1$. And then the other components, they just follow by the conservation. Yes, so, indeed, this is, again, this is a consistency check.

AUDIENCE: So for this case, there is no choice of quantization?

HONG LIU: Yeah, there's no choice for quantization, right. Actually, in AdS four, in some special dimensions, you can also do a different quantization. But the story is a little bit subtle. In some special dimensions, sometimes you can do it. Yeah, in AdS four, you can do a different boundary condition.

So now let me also mention the stress tensor, which is the other important one. And we mentioned that the stress tensor, which is due to the perturbation of the metric component. So if you write the AdS metric as some function times dz^2 plus $g_{\mu\nu} dx^\mu dx^\nu$, so this is the part of the metric parallel to the boundary, then we discussed, last time, if $g_{\mu\nu}$ goes to the boundary as something like this, then $\eta_{\mu\nu}$ plus some perturbation, then that means that, in the field theory side, you're turning on a stress tensor perturbation.

So that's what we argued last time. If the boundary value of your [INAUDIBLE] metric, say, is perturbed, away from the Minkowski metric, then corresponding from the field theory point of view, we turn on a source for the stress tensor.

So again, by using this relation, you can deduce that the stress tensor must have dimension, the corresponding operator must have dimension d . Again, this gives you exercise for yourself. And again, that is precisely the stress tensor. The dimension should satisfy that stress tensor. And again, because the stress tensor is a conserved current et cetera, always, no matter in what kind of series always have dimension d .

So now, after I have discussed this relation, let's now discuss how to compute the correlation functions. So let's discuss how to compute correlation functions. So the basic observable in the field theory, even more so in the conformal field theory, is just correlation functions of local operators. It's an important part of the [INAUDIBLE]. And in the field theory, typically, when we can see the correlation function of local operators, we use something called a generating function, though. I hope you're familiar with this.

Say, again, in the generating function, though-- so now let's consider Euclidean, which is a little bit simpler. So essentially, it's a Euclidean generating correlation function by adding such a term to your field theory. And if you want to calculate correlation functions of O , then just take derivative with ϕ and then satisfy equal to 0.

Then you get the correlation function of O . So I hope you are familiar with this. So here, you should think of ϕ and O as just some, really, I have a collection of all possible operators, even though I just write the single one. You should think of them as a collection of all operators. You can put anything here, stress tensor, conserved current, any scalar operator, any spin operator. Yeah, you can put anything there.

So in a simpler situation, you just don't turn on any operator. You just take ϕ equal to 0. So this is a simpler situation. Then you just have the partition function. So this correlation function is defined by the Euclidean path integral, for example.

And you just insert this in the Euclidean path integral. And when ϕ is equal to 0, this is just the Euclidean path integral. And this is just gives you a partition function of the whole system.

And since we believe the duality should be 2, then this partition function should be exactly the same as the partition function on the gravity side. Suppose you can compute the partition function from the gravity side. Then it must be the same. Otherwise, there's no equivalence.

So this is the partition function of the gravity side. So these have to be true. So now, with this relation, again, we are right in the heuristic form that some operator O is due to some [INAUDIBLE] ϕ . And the source for the O is due to the boundary value of ϕ -- again, I write in the heuristic form. So the boundary value of ϕ should be understood in that limit. Because of this, then that means we must have-- if we turn on a [INAUDIBLE] or a ϕ , deforming the series by this ϕ , the mass corresponding to the gravity side of the partition function, with this ϕ , have the boundary value given by ϕ .

Then it must be the same because we have identified turning on the boundary value to turning on the operator, turning on this thing in the field series. So we have now related the generating functional to the gravity partition function with some non-normalizable boundary conditions.

So, in general, give me one more minute. So in general, this equation is actually empty because we don't know how to define the gravity site. Full quantum gravity partition function. What are you talking about? If you don't know how to do that.

But we know how to compute this guy in the semiclassical limit, which is the string coupling goes to 0, and the other prime goes to 0 limit. So in this limit, we can just write the gravity path in function, essentially as the Euclidean path integral of all gravity fields.

Again, I'm using this heuristic form to write the boundary condition. But it should be understood in that form. And again, I'm by integrating over ϕ , you should imagine integrating of all gravity fields.

So now, in the limit, so this SE is proportional to $1/\kappa^2$, which is the Newton constant. So in the semiclassical limit, in this limit, the κ^2 goes to 0. So you can actually evaluate this path integral using the site point approximation.

So as the leading order, so this gravity can actually be evaluated. So leading order in the semiclassical limit, you can just evaluate it by site point approximation, then this is just given by SE. Evaluate it as a classical solution which satisfies the appropriate boundary conditions.

So that means, actually, in this limit, we can actually evaluate this explicitly. And the nice thing is, as we said before, the semiclassical limit actually corresponding to the [INAUDIBLE] limit in the field theory. So we can actually compute explicitly the generating functional in the [INAUDIBLE] of the limit over the field theory. So let's stop here.