HONG LIU: OK, let's start. So first let me just remind you what we did at the end of last lecture. So we see that the large N expansion of gauge theory have essentially exactly the same mathematical structure with, say, the mathematics of the [? N string ?] scattering. And so here the observable is a correlation function of gauging [? invariant ?] operators. And then these have a large N expansion as follows. And on this side you have just an N string scattering amplitude. Just imagine you have some kind of scattering of strings, with total number of N strings.

And then this also have expansion in terms of the string counting in this form. So now, if we identify-- so if we can identify the g string as 1/N. So if we identify g string with 1/N, then these two are essentially the same kind of expansion, OK? And you also can identify these external strings, string states, within the large N theory which we called the glueball states for single-trace operators.

And then each case is corresponding to [? sum ?] over the topology. It's an expansion [? in ?] terms of the topology. So here is the topology of the worldsheet string. And here is the topology of Feynman diagrams. Here is the topology of the Feynman diagrams.

So still at this stage, it's just like a mathematical correspondence. We're looking at two completely different things. But probably there's no-- yeah, no obvious connection between these two objects we are discussing. Yeah, we just have a precise mathematical structure. But one can actually argue that, actually, they also describe the same physical structure once you realize that when you sum over all possible Feynman diagrams. So once you realize that each Feynman diagram, say, of genus-h can be considered as a partition, or in other words, triangulization over genus-h surfaces, [? 2D ?] surfaces. OK.

So if you write more explicitly this fh, so if we write explicitly this fh, then this fh, this fnh, then will be corresponding to your sum of all Feynman diagrams of genus-h. Suppose G is the expression for each Feynman diagram. Say for each diagram.

And then I can just rewrite this. In some sense, I [? accept ?] all possible triangulation of [? a genus-g ?] surface. Say there will be some weight G. And summing over all possible triangulations of a surface is essentially-- so this is essentially the same as this sum over all possible surfaces.

So this is a discrete version. So sum all possible triangulations of some genus-g surfaces, or translations of genus-g surfaces. Then they can be considered as a discrete version of sum over all possible surfaces, OK?

AUDIENCE: So you're saying it's like a sum over [? syntheses, ?] like a simple [? x? ?]

HONG LIU: Exactly. Exactly. Yeah, because, say, imagine when you sum over surfaces, so you sum over all possible metric. You can put [INAUDIBLE]. And that's the same way as you sum over different discretizations of that surface once you have defined the unit for that discretization.

So if we can identify-- so for now record this Fh. So this Fh, this Fnh is the path integral over all genus-h surfaces with some string action, weighted by some string action. So if we can, say, identify this G with some string action-- the exponential of some string action. Then we would have-- then one can conclude that large N gauge theory is just a string theory, OK? That large N gauge theory is just a string theory, if you can do that.

In particular, the large N limits-- so large N limit here, as we discussed before, can considered as a classical theory of glueballs. Or a classical theory of the single-trace operators. So this would be matched to the classical string theory.

So as we mentioned last time, so I was mentioning before, this expression-- so just as in the as we discussed [INAUDIBLE], the [? expansion ?] in g string the same as expansion in the topology. And the expansion in the topology can also be considered as the expansion of the groups of a string. Because whenever you add a hole to the genus-- when you add the genus, and you actually add the string hole, you add the string loop diagram.

So in this sense, you can [? integrate ?] all these higher order corrections, as the quantum correction to this classical string behavior. So this is just a tree-level amplitude for string. And this [? goes ?] one into the loops. Whenever you add this thing, you add the loop.

OK. Is this clear? Now, remember what we discussed for the torus. If you've got a torus, then correspondingly you have a string split and joined together. And this split and join process you can also consider as a string loop, a single string going around a loop, [? just like ?] in the particle case, OK? In the standard field theory case.

And so the large N limit, which is the leading order term here, would map to a leading order in the string scattering. And the leading order in the string scattering-- they only consider tree-level [? skin ?] scatterings, and then corresponding to classical string theory.

And also the single-trace operator here can be mapped to the string states. Yeah, can be mapped to the string states. But this is only-- this is a very nice picture. But for many years, this was just a dream. And because this guy looks very different from this guy, but this is difficult. So this has some [? identification is ?] difficult for the following reasons.

So first, so this G just-- say your Feynman diagrams, amplitude for particular Feynman diagram. So G is typically expressed as product of field theory propagators. So imagine how you evaluate the Feynman diagram. The Feynman diagram, essentially, is just a product of the [? propagators. ?] And then you integrate it [INAUDIBLE] integrated over spacetime.

So they just take the Yang-Mills theory. And if you look at the expression for this diagram, of course, it looks nothing. So they look nothing like-- OK.

So let me make a few comments about this thing. Because if you want to match, say if I gave you a Yang-Mills theory, so I gave you a QCD, then you can write down-- then you can go to large N. You can write down expressions for the common diagrams. But if you say, I want to write it as a string theory, the first thing you have to say, what string theory do you want to compare?

So first you have to ask yourself what string action do you want to compare. So the string action, as we discussed last time, this describes the embedding of the worldsheet into some spacetime. OK, so this is worldsheet into a spacetime.

So this is also sometimes called the target space. So this is a spacetime. This string moves. And the mathematical of this is just the-- this is encoded in this mapping X mu sigma tau. OK, X mu is the coordinate for M. And then sigma tau is the coordinate as you parameterize your worldsheet.

So in order to write down action, of course, you have to choice of space manifold. You have to choose your spacetime. And also you have to-- when you fix the spacetime, you don't have a choice.

And sometimes the way to write down such kind of embedding is not unique. The action for such [? finding ?] is unique, so you only need to choose what action you include. And also often, in addition to this embedding, sometimes you can have additional internal degrees freedom. living on worldsheet. For example, you can have some fermions. Say if you have a superstring, then you can have some additional fermions are living on the worldsheet, in addition to this embedding.

So in other words, the choice of this guy in some sense is infinite. And without any clue-- so you need some clue to know what to compare the gauge theory to. And otherwise, even if this works, you're searching for some needle in the big ocean.

And then there's another very important reason why this is difficult, is that this string theory is formulated in a continuum. It's formulated in a continuum. And these Feynman diagrams, even if they're corresponding to some kind of string theory, they correspond into a discrete version of that.

So at best, it's a discrete version. So we expect such a geometric picture for G, for these Feynman diagrams, to emerge only at strong couplings. OK?

Emerge only at strong couplings for the following reason. So if you look at the Feynman diagram-- so the simplest Feynman diagram we draw before, say for example just this diagram. And if you draw it on the sphere, it separated the sphere into three parts, OK? So this [? discretizes ?] a sphere into three parts. And essentially, just as the sphere just becomes three points, because each particle is wanting to-- when you're trying to [INAUDIBLE] each part, you approximate it by one point. So essentially, in this diagram, you approximate the whole sphere essentially by three points. OK.

And of course, it's hard to see your [? magic ?] picture from here. And your [? magic picture ?] you expect to emerge, but your Feynman diagrams become very complicated. For example, if you have this kind of diagram, because of the four-point vertex. In principle, you can have all these diagrams.

And then this [INAUDIBLE] [? wanting ?] to discretize-- yes, I suppose this is on the torus. Suppose you have a-- for example, this could be a Feynman diagram on the torus, OK? For the vacuum [? energy. ?]

And now this is next some kind of proper discretization. And this will go to a continuum limit, say when the number of these box go to infinity. When the number of box go to infinity, then you need a number of propagators, and the number of vertices goes to infinity, OK? So in order for continuum, a picture to emerge, so you want those complicated diagrams-- it's not your number of vertices or large number of propagators that dominate.

And for those things that dominate, then you need the strong coupling. Because with this coupling, this is the leading order diagram. And there's no geometry from here, OK? So in order to have the geometry, you want the diagram are very, very complicated, so that they really-- [INAUDIBLE] a triangulation of a surface. A weak coupled diagram with small number of lines will cause [? one ?] [? and two ?] are very close triangulization of a surface. So we expect this only appears in strong couplings, OK? Yeah.

AUDIENCE: By the cases like we have to sum over all the [INAUDIBLE].

HONG LIU: Yeah, sum over the [INAUDIBLE] diagram.

AUDIENCE: Including those simple ones.

HONG LIU: Including those simple ones. So that's why you want to-- so if you're in a weak coupling, then the simple ones-- so we sum all those diagrams. And each diagram you can associate with a coupling power. So at weak coupling, then the lowest order term would just dominate. And the lowest order term have a very simple diagrams. And then that's because [? one ?] and [? two ?] are very crude triangulization over the surface.

But if you have a strong coupling-- in particular, if you have an infinite coupling-- the diagrams, the infinite number of vertices will dominate. And then that's because [? one ?] and [? two ?] have very fine triangulization over the surface. And then that can go to the [INAUDIBLE].

AUDIENCE: [INAUDIBLE] interaction a coupling constant has been [? dragging ?] out from--

HONG LIU: No. That's just N dragged out.

AUDIENCE: Oh, I see.

HONG LIU: No, there's what we call this [INAUDIBLE] still remaining. By coupling, it's only [? N. ?]

AUDIENCE: [INAUDIBLE]

HONG LIU: No, no, this isn't to [? hold ?] coupling. In coupling we mean that [INAUDIBLE]. So example we talk about, [? because one ?] [? and two ?] [INAUDIBLE]. Yeah, and then we make more precise. So in the [? toy ?] example we talked about before. So previously we talked about this example, N divided by lambda, trace, say 1/2 partial phi squared, plus 1/4 phi to the power 4. And strong coupling means the lambda large. Because of the N I've already factored out, so you're coupling just lambda.

AUDIENCE: Oh, I see.

HONG LIU: Yes.

AUDIENCE: So in these [INAUDIBLE] the propagator in that version would become the spacetime integration?

HONG LIU: Hm?

AUDIENCE: I was just wondering how the propagator can [? agree, ?] can match to the spacetime [INAUDIBLE].

HONG LIU: Yeah, yeah. So the propogator-- yeah, propagator you do in the standard way. You just write down your propagator, and then you try to repackage that. As the question, you said, whatever your rule, Feynman rule is we just do that Feynman rule. And you write down this expression. It's something very complicated. And then you say, can I find some geometric interpretation of that?

Yeah, what I'm saying is that doing from this perspective is very hard because you don't know what thing to compare. And further, in the second, you expect that your [INAUDIBLE] would emerge only in those very complicated diagrams. And those complicated diagrams we don't know how to deal with. Because they only emerge in the strong coupling limit, but in the strong coupling limit, we don't know how to deal with that. And so that's why it's also difficult.

But [? nevertheless, ?] for some very simple theories, say, if you don't consider the Yang-Mills theory, you don't consider the gauge theory. But suppose you do consider some matrix integrals. Say, for very simple systems, like a matrix integral. So this structure emphasizes-- this structure only have to do with you have a matrices, OK? And then you can have matrix-valued fields [? or ?] this structure will emerge. Or you only have a matrix integral. So there no field at all, just have a matrix integral. That same structure will also emerge.

For example. I can consider theory-- have a theory like this. Something like this. And have a theory like this, OK? And M is just some [INAUDIBLE] matrices. So this is just integral. And the same structure will emerge, also, in this series when we do large N expansion.

So that structure have nothing to do-- yeah, you can do it. So matrix integral is much simpler than [INAUDIBLE] field theory because you have much less degrees freedom. So for simple systems like, say, your matrix integral or matrix quantum mechanics, actually, you can guess the corresponding string theory. Because also the string theory in that case is also very simple. You can guess where is simple string theory. But it's not possible for field theory. It's not possible for field theory. Yes.

AUDIENCE: So what do you mean by matrix quantum mechanics? Like that, OK.

HONG LIU: So this is a matrix integral. And I can make it a little bit more complicated. So I make this M to depend on t, and then this become a matrix quantum mechanics. Say trace M dot squared plus M squared plus M4. Then this become a matrix quantum mechanics, because it only have time.

And then I can make it more complicated. I can make M be t, x. Then this becomes one plus one dimension of field theory.

AUDIENCE: So in what context is this matrix quantum mechanics [? conflicted? ?]

HONG LIU: Just at some [? toy ?] model. I just say, and this is a very difficult question. You said, I don't know how to deal with field theories. Then this [? part of it's ?] a simple system. And then just try to use this philosophy, can see whether it can do it for simple system. And then you can show that this philosophy actually works if you do a matrix integral or matrix quantum mechanics.

Simple enough, matrix integral and matrix quantum mechanics. OK. And if you want references, I can give you references regarding these. There's a huge, huge amount of works, thousands of papers, written on this subject in the late '80s and early '90s.

So those [? toy ?] examples just to show actually this philosophy works. I just showed this philosophy works, OK? But it's not possible if we want to go to higher dimensions. Actually, there's one paper-- let me just write it here. So this one paper explains the philosophy.

So here I did not gave you many details, say, how you write this G down, how you in principle can match with this thing. With [? another ?] maybe [INAUDIBLE] you can make this discussion a little bit more explicit, but I don't have time. But if you want, you can take a look at this paper.

So this paper discusses the story for the matrix quantum mechanics. But in the section 2 of this paper-- so this is a paper by Klebanov. So in the section 2 of this paper, it explains this mapping of Feynman diagrams to the string action. And this discretization picture give you a nice summary of that philosophy with more details than I have given to you. So you can take a look at that. And this paper also has some other references if you want to take a look at it. OK. Any questions? Yes.

AUDIENCE: Sorry, but who was the first to realize this connection between the surfaces in topology of Feynman diagrams?

HONG LIU: Sorry?

AUDIENCE: Who first realized this relation between topology and--

HONG LIU: So of course, already when 't Hooft invented this large N expansion, he already noticed that this is similar to string theory. So he already commented on that. And he already commented on that. And for many years people did not make progress. For many years, people did not make progress. But in the late '80s-- in the mid to late '80s, people started thinking about the question from this perspective, not from that perspective.

So they started to think about the order from this perspective. Because just typical string theory are hard to solve, et cetera. So people think, maybe we can actually understand or generalize our understanding of string theory by discretize the worldsheets. And then they just integrate over all possible triangulization, et cetera.

And then they realized that that thing actually is like something over Feynman diagrams. And then for the very simple situations, say like if you have only a matrix integral, actually you can make the connection explicit. So that was in the late '80s. So people like [? McDowell ?] or [? Kazakov ?] et cetera that were trying to explore that. Other questions?

AUDIENCE: I'm having trouble seeing how the sum over all triangulations [INAUDIBLE] each surfaces. How does that correspond to the discrete version of summing over all [INAUDIBLE]?

HONG LIU: Right.

AUDIENCE: That's the discrete sum over all possible [? genus-h, ?] right?

HONG LIU: Yeah. I think this is the example. Yeah, let's consider torus. So a torus is a box with this identified with this, and this identified with that. OK. And let me first just draw the simplest partition here. Just draw like that.

Yeah. Let me just look at these two things. So suppose I give each box-- so if I specify each box, say, give a unit area. OK? And I do this one, I do that one, or I do some other ways to triangulize it.

Then because [? one and two ?] give a different symmetric to the surface. And then because [? one and two ?] integrate over all possible metric on this surface. And they integrate over all possible metric on this surface, you can integrate [INAUDIBLE] all possible surfaces.

AUDIENCE: In the case of the strings for example, [? we put some ?] over the torus here and the torus and the torus there.

HONG LIU: No, no. You only sum over a single torus. Now, what do you mean by summing over torus here, torus there?

AUDIENCE: I thought like in the path integral, in the case of the string theory--

HONG LIU: No, you're only summing over a single torus. You're only summing over a single surface, but all possible ways to write-- all possible ways to draw that surface.

So what you said about summing torus here, summing torus there, because [INAUDIBLE] what we call the disconnected amplitudes. And then you don't need to consider them in physically disconnected amplitude. You can just [? exponentiate ?] what we call by connected amplitude. And you don't need to do that separately. So once you know how to do a single one, and the disconnected one just automatically obtained by [? exponentiation. ?]

AUDIENCE: [INAUDIBLE]

HONG LIU: Sorry? No, no. Here the metric matters, the geometry matters. It's not just the topology.

AUDIENCE: [INAUDIBLE] Feynman diagram [INAUDIBLE]?

HONG LIU: Yeah. Yeah, just the key is that the propagator of the Feynman diagram essentially [? encodes ?] the geometries. And in encoding a very indirect way. Yeah. Just read this part. This section only have a few pages, but contain a little bit more details on what I have here. It requires maybe one more hour to explain this in more detail.

Yeah, this is just that. I just want to explain this philosophy. I don't want to go through the details of how you do this. OK, good. So now let me just mention a couple of generalizations.

So the first thing you already asked before, I think maybe both you have asked. Let me just mention them quickly. And if you are interested, I can certainly give you a reference for you to read about them, or I can put it in [? your P ?] sets.

And so, so far, it's all matrix-valued fields, OK? But if you can see the theory-- or in other words, in the mathematical language, say, it's an adjoint representation. It's an adjoint representation of the-- because our symmetries are UN, it's a UN gauge group. OK? UN gauge group.

But you can also, for example, in QCD, you also have quarks. So you also have field in the fundamental representations. So it can also include field in the fundamental representation. So rather than matrix-valued, they're N vector. OK, they're N [? vectors. ?]

So for quarks, of course, for the standard QCD N will be 3, so you have three quarks. You have three different colored quarks. And so then your Feynman diagrams, in addition to have those matrix [? lines, ?] which you have a double line. And now here you only have a single index, OK? And then you only have a single line. So the propagator of those quarks will just have a single line. And then also in your Feynman diagram you can have loops over the quarks, et cetera.

So you can again work this out. And then you find it is a very nice large N expansion. And then you find the diagrams, the Feynman diagrams. Now you find in this case the Feynman diagrams can be classified by 2D surfaces with boundaries.

So essentially, you have-- and let me just say, for example, this is the vacuum diagrams, for all the vacuum process. Then you can [INAUDIBLE] or the vacuum diagrams. And then they can all be [? collectified. ?] So previously, we have a matrix-valued field. Then all your vacuum diagrams, they are corresponding closed surfaces-- so sphere, torus, et cetera. But now if you include the quarks, then those surfaces can have boundaries. And then [INAUDIBLE] into the quark groups, et cetera. And then they [? cannot ?] be classified.

And so these also have a counterpart if you try to map to the string theory. So this [INAUDIBLE] [? one and ?] [? two, ?] string theory. There's string theory with both closed and open strings. And so essentially those boundaries give rise to the open strings. So here, it's all closed strings. It's all closed surface. Well, now you can, by adding the open strings, and then you can, again, have the correspondence between the two. OK. So all the discussion is very similar to what we discussed before. We just apply all this the same philosophy to the quarks. Yes.

AUDIENCE: [INAUDIBLE] do the same trick on string theory and find some sort of expression which then will map to some higher order surfaces, [INAUDIBLE]?

HONG LIU: Sorry, say that again?

AUDIENCE: [INAUDIBLE] Feynman diagrams we move to string theory for surfaces. Is there some [INAUDIBLE] from surfaces just they go one more [? step up? ?]

HONG LIU: You mean higher dimensions, not strings. Yeah, that will become-- of course, that's a [? lateral ?] idea. So that will [INAUDIBLE] you can consider [? rather ?] strings, you can consider two-dimensional surface, a two-dimensional surface moving in spacetime.

And then [INAUDIBLE] into [? so-called ?] the membrane theory. But let's say where it turns out to be-- turns out string is a nice balance. It's not too complicated or not too simple. And it give you lots of structure. But when you go to membrane, then the story become too complicated, and nobody knows how to quantize that theory.

So the second remark is that here we consider UN. So here our symmetry group is UN. Because our phi-- phi there is [? commission. ?] So when you have a [? commission ?] matrix, then there's a difference between the two indices, so we put one up and one down. So they are propagators that lead to-- so it leads to the lines with arrows, because we need to distinguish upper and lower indices.

OK? Between the two indices. But you can also consider, for example, phi is a symmetric matrix. Say it's a real symmetric matrix. It's a real symmetric, or real anti-symmetric. In those cases, then there's no difference between the two indices.

And then when you draw a propagator-- so in this case the symmetry group would be, say, SON, say, or SPN, et cetera. And then the propagators, they will no longer have orientations. OK? They will no longer have orientations. Because you can no longer-- yeah.

So this will give rise-- so let me write it closer. So this will give rise to unorientable surfaces. Say, for example, to classify the diagrams, you can no longer just use the orientable surfaces.

You also have to include the non-orientable surfaces to classify the diagrams.

And the [INAUDIBLE] this also have a precise counterpart into unorientable strings. No, non-orientable strings. Yeah, I think non-orientable, non-orientable surfaces. Also non-orientable strings.

Good. So I'm emphasizing how difficult it is if, say, we want to start with QCD and then try to find the string theory description. But this still, [? none of ?] this tries-- I just try.

OK, so let's just consider, just take large N generalization of QCD. So this, again, will be some UN gauge theory, UN Yang-Mills theory, say, in 3 plus 1 dimensional Minkowski spacetime. And can we say anything about its string theory description? So [INAUDIBLE]. So maybe it's difficult, but let's try to guess it.

OK. So in physics, in many situations, a seemingly difficult problem, if you know how to guess it, actually you can get the answer. On, for example, quantum hole effects, fractional quantum hole effects, you can just guess the wave function. So of course, the simplest guess-- so this is some gauge theory in 3 plus 1 dimensional Minkowski spacetime.

So now we say this is a string theory. So natural guess is that this maybe is a string theory, again, in the 3 plus 1 dimensional Minkowski spacetime. OK? So we just take what-- so these will, of course, run into a string, propagating in this spacetime, OK?

As I said, when you write down the string theory, you first have to specify your target space, which, as the string moves, the larger question would be just, should it be the gauge theory's Minkowski spacetime. Maybe this string theory should be. OK?

And then this. Then you can just try to-- then you can just write down the simplest action. So maybe say Nambu-Goto action, which we wrote last time, OK? Or the [? old ?] Polyakov action. So this Nambu-Goto action will result [INAUDIBLE] Polyakov. And let me not worry about that. For example, you can just guess, say, maybe this is a string theory also in the Minkowski spacetime. Say, consider the simplest action. Or the equivalent of this, OK?

Then at least what you could try-- now you actually have an action. Now you think that you have this object. Now you think you can compare. OK, now you can essentially compare. Say, in QCD you calculated your Feynman diagrams, and now just compare.

But of course, you still have the difficulty. Of course, you have to go to strong coupling to see the geometric limit, et cetera. But in principle, it's something you can do.

But this actually does not work. OK? This does not work, for the following simple reason. Firstly, that such a string theory-- so a string theory, actually the remarkable thing about the string is that if you have a particle, you can put the particle in any spacetime. But strings are very picky. You cannot put them in any spacetime. And they can only propagate consistently, quantum mechanically consistently, in some spacetime but not in others.

So for example, if you want to put the string to propagate in this 3 plus 1 dimensional Minkowski spacetime, then you actually find that the theory is mathematically inconsistent. So such a string theory is inconsistent. It's mathematically inconsistent. Except for the D equal to 26 or 10. OK? So 26 if you just purely have the theory, and 10 if you also add some fermion.

So such a string theory does not exist mathematically. So you say, oh, OK. You say, I'm a smart fellow. I can go around this. Because we want the Minkowski spacetime. Because those gauge theory propagating the Minkowski spacetime, so this Minkowski [INAUDIBLE] must be somewhere. They cannot go away, because all these glueballs [INAUDIBLE] in this 3 plus 1 dimensional Minkowski spacetime. And if we want to identify the strings with those glueballs, those strings must at least [? know ?] some of this Minkowski spacetime.

And then you say, oh, suppose you tell me that this string theory is only consistent in 10 dimension. But then let me take a string theory in 10 dimensions, which itself consistent. But I take this 10-dimensional spacetime to have the form of a 3 plus 1 dimensional Minkowski spacetime. And the [? time, ?] some compact manifold, OK? Some compact manifold.

And in such case-- so if this is a compact manifold, then the symmetry of this spacetime, so the spacetime symmetry still only have the 3 plus 1 dimensional, [? say, ?] Poincare symmetry. Because if you want to describe the QCD in 3 plus 1 dimension, QCD has the Poincare symmetry. You can do Lorentz transformation, and then you can do rotation. Or you can do translation. The string theory should not have more symmetries or less symmetries than QCD. They should have the same symmetries because they are supposed to be the same description.

But if you take the 10-dimensional Minkowski space, of course, it's not right. Because the 10-dimensional Minkowski space have 10-dimensional translation and 10-dimensional Lorentz symmetry. But what you can do is that you take this 10-dimensional space to be a form of the 3 plus 1 dimensional Minkowski spacetime and times some additional compact manifold, and then you have solved the symmetry problem.

But except this still does not work because the string theory, as we know, always contain gravity. And if you put a string theory on such a compact space N, [? there would be ?] always leads to a massless spin-2 particle in this 3 plus 1 dimensional part. But from Weinberg-Witten theorem we talked in the first lecture, in the QCD you are not supposed to have a 3 plus 1 dimensional massless spin-2 particle, OK? And so this won't work. So this won't work. Because this contains--

In 3 plus 1 dimensional [? Minkowski space, ?] which does not have-- OK? Or in the large N [INAUDIBLE]. So this does not work. So what to do? Yes?

AUDIENCE: So does this just mean that it's mathematically inconsistent?

HONG LIU: No, no. This does not mean it is mathematically inconsistent. It just means this string theory cannot not correspond to the string theory [? describe ?] QCD. The string theory description-- the equivalent string theory for QCD cannot have this feature. Yeah, just say this cannot be the right answer for that string theory. This string theory is consistent. Yes.

AUDIENCE: So is that you were saying if there is a massless spin-2 particle in that string theory, there has to be a [? counterpart in the ?] QCD.

HONG LIU: That's right.

AUDIENCE: If there is not a [INAUDIBLE], that won't work

HONG LIU: Yeah. This cannot be a description of that. From Weinberg-Witten theorem, we know in QCD there's no massless spin-2 particle. Yes.

AUDIENCE: I thought we have talked about maybe we can do strings to [? find ?] QCD in a different dimension [? in ?] space.

HONG LIU: We will go into that. But now they are in the same dimension, because this Minkowski 4, this will have-- because this is a compact [? part, ?] it doesn't matter. So in this part, [? there are ?] massless spin-2 particles. This does not [? apply ?] in QCD.

So what can you do? So most people just give up. Most people give up. So other than give up, the option is say maybe this action is too simple. Maybe you have to look at more exotic action. OK. So this is one possibility. And the second possibility is that maybe you need to look for some other target space. OK.

But now, what if you go away from here? Once you go away from here, everything else is now becoming such little in the ocean, because then you don't have much clue what to do. We just say, your basic guess just could not work. So for many years, even though this is a very intriguing idea, people could not make progress.

But now we have hindsight. But now we have hindsight. So we know that even this maybe cannot be described by a four-dimensional-- so even though this cannot have a-- so this cannot have a massless spin-2 particle in this 3 plus 1 dimension of Minkowski spacetime. Maybe you can still have some kind of graviton in some kind of a five-dimensional spacetime. You have some five dimensions, in a different dimension.

So there were some rough hints. Maybe you can consider there's a five-dimensional string theory. So let me emphasize when we say five or four, I always mention the non-compact part. So the compact part, it doesn't count because compact part just goes for the ride. What determines the properties, say, of a massless particle, et cetera, is the uncompact of the spacetime.

Yeah, because this is a 10-dimensional spacetime. This is already not [INAUDIBLE]. So maybe we [? change ?] for string theory in five-dimensional uncompact.

AUDIENCE: Five, so in 4 plus 1?

HONG LIU: Yeah. In 4 plus 1 uncompact spacetime. Yes.

AUDIENCE: [INAUDIBLE] compactors. When you say compact, do you mean the mathematical definition of compactness?

HONG LIU: Yeah, that's right. Yeah, I just say there is a finite volume. Just for our purpose here, we can do it simply. Just let's imagine-- yeah, compact always has a finite volume, for example. Yes?

AUDIENCE: Why can we just ignore the compact dimensions? Is there any condition on how big they're allowed to be or something, like limit?

HONG LIU: Yeah, just when you have-- so if you know a little bit about this thing called the Kaluza-Klein theory. And you know that the compact part-- the thing is that if you have a theory [? based ?] on uncompact and the compact part, and then most of the physical properties is controlled by the physics of uncompact parts. And this will determine some details like the detailed spectrum, et cetera. But the kind of thing we worry about, whether you have this massless spin-2 particle, et cetera, will not be determined by this kind of thing.

AUDIENCE: Is there any volume limit on the compact part, like maximum?

HONG LIU: No, it's fine to have a finite volume.

AUDIENCE: Just finite, but can it be large?

HONG LIU: No matter how large, this have infinite. It's always much smaller than this one. Yeah, but now it's just always relative. It's always relative. Yes.

AUDIENCE: Tracking back a little, is there any quick explanation for 26 and 10 are special, or is it very complicated?

HONG LIU: Um.

[LAUGHTER]

No, it's not complicated. Actually, we were going to do it in next lecture. Yeah, next lecture we will see 26, but maybe not 10. 10 is little bit more complicated. Most people voted for my option one, so that means you will be able to see the 26. Right.

AUDIENCE: Who [? discovered ?] 26 and 10? I mean, they are specific for this [INAUDIBLE] action rate, so for other action would be something else.

HONG LIU: Specifically for the Nambu-Goto action is 26. And for the 10, you need to add some additional fermions and make it into a so-called superstring, then become 10. And even this 26 one is not completely self-consistent. And anyway, there's still some little, tiny problems with this. Anyway, so normally we use 10.

OK so now, then there's some tantalizing hints for the-- say, maybe you cannot do it with the 3 plus 1 dimensional uncompact spacetime. Maybe you can do a 4 plus 1 dimensional uncompact. So the first is the holographic principle, where you have length. Holographic principle we have learned because there we say, if you want to describe a theory with gravity, then this gravity should be able to be described by something on its boundary.

And the string theory is a theory with gravity. So if the string theory should be equivalent to some kind of QCD, some kind of gauge theory without gravity, and then from holographic principle, this field theory maybe should be one lower dimension, OK? In one lower dimension. Is the logic here clear?

AUDIENCE: Wait, can you say that again?

HONG LIU: So here we want to equate large N QCD with some string theory. But string theory we know contains gravity. A list of all our experience contain gravity. But if you believe that the gravity should satisfy holographic principle, then the gravity should be equivalent, according to holographic principle, gravity in, say, D dimensional spacetime can be described by something on its boundary, something one dimension lower.

AUDIENCE: But I thought the holographic principle was a statement about entropy.

HONG LIU: No, it's a state started from a statement about entropy. But then you do a little bit of leap. So what I call it little bit of a conceptual leap is that the-- or [? little ?] leap of faith is that you promote that into the statement that said the number of degrees freedom you needed to describe the whole system. Yeah, so the holographic principle is that for any region, even the quantum gravity theory, for any region, you should be able to describe it by the degrees of freedom living on the boundary of that region. And degrees freedom living on the boundary of that region, then it's one dimensional lower.

AUDIENCE: Wait, so can I ask one question about that? If I have some region, some volume in space, some closed ball or something. And I live in a universe which is, for example, a closed-- like maybe they live on some hypersphere or something like this. Then how do I know whether I'm-- how do I know that the information is encoded? How do I know whether I'm inside the sphere or outside of the sphere?

For example, we see that the entropy that has to do with the sphere basically tells you about how much information can you contain inside the sphere. But if you live in a universe which is closed or something, then you don't know whether you're inside or outside the sphere.

HONG LIU: Yeah, but that's a difficult question. Yeah, if you talk about closed universe here, we are not talking about closed universe.

AUDIENCE: I see.

HONG LIU: Yes.

AUDIENCE: I thought the holographic principle is that the number of degrees freedom inside the region is actually bounded by the area.

HONG LIU: Right, it's bounded by--

AUDIENCE: Yeah, but why is it that we use that degree of freedom living on the boundary?

HONG LIU: There are several formulations of that. First is that the total number of degrees freedom in this region is bounded by the area. And then you can go to the next step, which is maybe the whole region can be just described by these degrees of freedom living on the boundary on that region.

AUDIENCE: Is that because, say, the state of density on the boundary [INAUDIBLE] the state on the boundary is proportional to the area of the boundary?

HONG LIU: Yeah. Exactly. That's right.

AUDIENCE: So here our goal is to recover the large N theory in 3 plus 1 dimensions without gravity. So we have no gravity. You can't 3 plus 1.

HONG LIU: Right. Yeah, so if that is supposed to be equivalent to the gravity theory, and the gravity [? theory ?] to find the holographic principle, and then the natural guess is that this non-gravitational field theory should live in one dimensional lower. OK? So this is one hint.

And the second is actually from the consistency of string theory itself. So this is a little bit technical. Again, we will only be able to explain it a little bit later, when we talk about more details about string theory. You can [? tell, ?] even though the string theory in this space is inconsistent. But there's a simple way. This is-- it's not a simple way.

So what's happening is the following. So if you consider, say, a string propagating in this spacetime, and there are some symmetries on the worldsheet. And only in the 10 and 26 dimension, those symmetries are satisfied quantum mechanically. And in other dimensions, those symmetries, somehow, even though classically it's there, but quantum mechanically it's gone. And those symmetries become-- because they are gone quantum mechanically, then it leads to inconsistencies.

And it turns out that there's some other way you can make that consistent, to make that symmetry still to be valid, is by adding some new degrees of freedom. OK? It's just there's some new degrees freedom dynamically generated. And then that new degrees freedom turned out to behave like an additional dimension. OK. Yeah, this will make no sense to you. I'm just saying a consistency of string theory actually sometimes can give you one additional dimension.

AUDIENCE: What is the difference between these inconsistencies, talking about anomalies and--

HONG LIU: It is anomalies. But here it's called gauge anomalies. It's gauge anomalies is at the local symmetry anomalies, which is inconsistent.

AUDIENCE: So just-- maybe this is not the time to ask this-- but are the degrees of freedom that you need to save you from this inconsistency problem. So do they have to be extra dimensions of space? Or what I'm saying is that if we need to do string theory in 10 dimensions, is it really four dimensions plus six degrees of freedom? Or are they actually six bona fide spatial dimensions?

HONG LIU: Oh, this is a very good question. So if you have-- yeah, this something we would be a little bit more clear just even in-- oh, it's very late. Even the second part of this lecture is that here you have four degrees of freedom, you have six degrees of freedom.

But turns out, if you only consider this guy, then this four degrees freedom by itself is not consistent. It's [? its own ?] violation of the symmetry at the quantum level. And then you need to add more, and then one more, because of course one and two have extra dimension. Anyway, we can make it more explicit in next lecture. Here I just throw a remark here.

Anyway, this guy-- this is purely hindsight. Nobody have realized this point, this first point, nobody have realized it before this holographic duality was discovered. Nobody really made this connection. And at this point, saying there should be a five-dimensional string theory describing gauge theory, that was made just before the discovery. I will mention that a little bit later.

Anyway, so now let's-- let me just maybe finish this, and we have a break. So now let's consider-- suppose there is a five-dimensional spacetime, string theory in some five-dimensional spacetime, say 4 plus 1 dimensional spacetime that describes QCD. Then what should be the property of this Y? So this Y denotes some manifold Y. OK?

So as I mentioned, it must have at least all the symmetries of the QCD, but not more. Should have exactly the same amount of symmetries. So that means it must have the translation and

Lorentz symmetries of QCD. OK?

So that means the only metric I can write down must be of this form. The only metric I can write down, the metric must be have this form. So this az just some function. And z is the extra dimension to a Minkowski spacetime. And this is some Minkowski metric for 3 plus 1 dimension.

AUDIENCE: You mean it's like a prototype to four dimension, we have to get the Minkowski space.

HONG LIU: Yeah. Just say whatever this space, whatever is the symmetry of this-- so the symmetry of this spacetime must have the Poincare-- must have all the symmetry of the 3 plus 1 dimensional Minkowski spacetime. Then the simplest way, you're saying that the only way to do it is just you put the Minkowski spacetime there as a subspace. And then you have one additional space, and then you can have one additional dimension.

And then, because you have to maintain the symmetries and [INAUDIBLE] to be thinking then you can convince yourself that the only additional degrees freedom in the metric [INAUDIBLE] is the overall function. So the function of this z, and nothing else. OK.

AUDIENCE: Can that be part of kind of a scalar in Minkowski space?

HONG LIU: Yeah. Let me just say, this is most general metric, consistent with four dimensional, 3 plus 1 dimensional, Poincare symmetries.

AUDIENCE: Why this additional dimension always in a space part? Can it be in a time-like part? Like a 3 plus 2?

HONG LIU: Both arguments suggest it's a space part. So because this is just the boundary of some region there's a spatial dimension [? reduction ?], not time. So is this clear to you? Because you won't have a Minkowski spacetime, so you must have a Minkowski here.

And then in the prefactor of the Minkowski, you can multiply by anything, any function, but this function cannot depend on the X. It can only depend on this extra dimension. Because if you have anything which depend on capital X, then you have violated the Poincare symmetry. You have violated the translation [? X. ?]

So the only function you can put before this Minkowski spacetime is a function of this additional dimension. And then by redefining this additional dimension, I can always put this overall factor in the front. Yeah, so this tells you that this is the most general metric. OK? So if it's not clear to you, think about it a little bit afterwards.

So these are the most as you can do. So that's the end. So you say, you cannot determine az, et cetera. So this is as most you can say for the QCD. But if the theory, if the field theory is scale invariant, say, conformal field theory, that normally we call CFT, OK? So conformal field theory.

Then we can show this metric. So let me call this equation 1. Then 1 must be [INAUDIBLE] spacetime.

AUDIENCE: [INAUDIBLE] symmetry on the boundary as well, [INAUDIBLE]?

HONG LIU: Yeah, I'm going to show that. So if the field theory is scale invariant, that means that the fields theory have some additional symmetry, should be satisfied by this metric. And then I will show that this additional scaling symmetry will make this to precisely a so-called anti-de Sitter spacetime.

AUDIENCE: Field theory, and then the 3 plus 1.

HONG LIU: Yeah. Right. If the field theory, say the-- QCD does not have a scale. It's not scaling right, so I do not say a QCD anymore. Just say, suppose some other field theory, which have large N expansion, which is also scale invariant. And then the corresponding string theory must be in anti-de Sitter spacetime.

AUDIENCE: Are we ever going to come back to QCD, or is that a--

HONG LIU: No, that's it. Maybe we'll come back to QCD, but in a somewhat indirect way. Yeah, not to your real-life, beloved QCD.

AUDIENCE: So no one's solved that problem still?

HONG LIU: Yeah, no one's solved that problem yet. So you still have a chance. So that remains very simple. So let me just say, then we will have a break. Then we will be done. I think I'm going very slowly today.

So scale invariant theory-- is invariant under the scaling for any constant, constant lambda. So scale invariant theory should be invariant under such a scaling. And then now we want to require this metric also have this scaling. OK? So now, we require 1 also have such scaling.

That's scaling symmetry.

OK, so we just do a scaling X mu go to lambda X mu. And then this term will give me additional lambda squared. So we see, in order for this to be the same as before, the z should scale the same, OK? So in order for this to be-- so we need z to scale as the same, in order I can scale this lambda out.

After I scale this lambda out, I also need that a lambda z should be equal to 1 over lambda az, OK? So the scaling symmetry of that equation requires these two conditions. So on the scaling of z, this a lambda z should satisfy this condition. Then the lambda will cancel.

So this condition is important because we did scale them homogeneously. Otherwise, of course, lambda will not drop out. And the second condition just makes sure lambda is canceled. OK, is it clear?

So now this condition just determined that az must be a simple power, must be written as R divided by z. See, R is some constant. And now we can write down the full metric. So now I've determined this function up to our overall constant. So the full metric is dS square equal to R squared divided by z squared dz squared plus eta mu, mu, dX mu, dX mu.

And this is precisely AdS metric, written in certain coordinates. And then this R, then you adjust the curvature radius of AdS. So if you don't know about anti-de Sitter spacetime, it doesn't matter. So this is the metric, and the name of this metric is anti-de Sitter. And later we will explain the properties of the anti-de Sitter spacetime.

So now we find, so now we reach a conclusion, is that if I have a large N conformal field theory in Minkowski D-dimensional space, time. So this can be applied to any dimensional. It's not necessary [? to be ?] 3 plus 1. In D-- so this, if it can be described by a string theory, should be string theory in AdS d plus 1. And in particular, the 1/N here is related to the g strings here, the string coupling here. So this is what we concluded. Yes?

AUDIENCE: So all we've shown is that there is no obvious inconsistency with that correspondence.

HONG LIU: What do you mean there's no obvious?

AUDIENCE: As in, we didn't illustrate any way that they--

HONG LIU: Sure, I'm just saying this is a necessary condition.

AUDIENCE: Right, so at least that is necessary.

HONG LIU: Yeah, this is a necessary condition. So if you can describe a large N CFT by our string theory-- and it should be a string theory-- yeah, this proposal works. This proposal passed the minimal test.

AUDIENCE: I have a question. So when Maldacena presumably actually did figure this out, you said that this resulted from the holographic principle, like it was just figured out right before he did it. Was he aware of the holographic--

HONG LIU: No, here is what I'm going to talk. So Maldacena, in 1997, Maldacena found precisely-- in 1997, Maldacena found a few examples of this, precisely realized this. And not using this mass or using some completely indirect way, which we will explain next. So he found this through some very indirect way. But in principle, one could have realized this if one kept those things in mind.

So now let me tell you a little bit of the history, and then we will have a break. Then we can go home. It depends on whether you want a break or not. Maybe you don't want a break.

Yeah, let me tell you a little bit of history. So yeah, just to save time, let me not write it down, just say it. So in the late '60s to early '70s, so string theory was developed to understand strong interactions. So understanding strong interactions was the problem. At the time, people were developing string theory to try to understand strong interactions.

So in 1971, our friend Frank, Frank Wilczek, and other people, they discover the asymptotic freedom. And they established the Yang-Mills theory as a description of strong interaction which now have our QCD. And so that's essentially eliminated the hope of string theory to describe QCD.

Because the QCD seems to be very different. You [? need ?] the help of string theory to describe strong interaction because the QCD [INAUDIBLE] gauge theory, it's very different from the string theory. So people soon abandoned the string theory.

So now we go to 1974. So 1974, a big number of things were discovered in 1974. So 1974 was a golden year. So first is 't Hooft realized his large N expansion and then realized that this actually looks like string theory. And then completely independently, Scherk, Schwarz, and [? Yoneya, ?] they realized that string theory should considered a theory of gravity, rather than a theory of strong interaction.

So they realized actually-- it's ironic, people started doing string theory in the '60s and '70s, et cetera. But only in 1974 people realized, ah, string theory always have a gravity and should be considered a theory of gravity. Anyway, so in 1974, they realized the string theory should be considered as a gravity.

So that was a very, very exciting realization, because then you can have [? quantum ?] gravity. But by that time, people had given up on string theory. So nobody cared about this important observation. Nobody cared about this important observation.

So, also in the same year, in 1974, Hawking discovered his Hawking radiation. And they established that black hole mechanics is really a thermodynamics. Then really established that the black hole is a thermodynamic object,

And in 1974 there's also a lot of important discovery-- which is related to MIT, so that's why I'm mentioning it-- is that people first really saw quarks experimentally, is that, again, our friend, colleague Samuel Ting at Brookhaven, which they discovered a so-called charmonium, which is a bounce state of the charm quark and the anti-charm quark.

And because the charm quark is very heavy, so they form a hydrogen-like structure. So in some sense, the charmonium is the first-- you first directly see the quarks. And actually, even after the 1971, after asymptotic freedom, many people do not believe QCD. They did not believe in quarks. They say, if there's quarks, why don't we see them?

And then in 1974, Samuel Ting discovered this charmonium in October. And so people call it the October Revolution.

[LAUGHTER]

Do you know why they laugh? OK. Anyway. Yeah. Yeah, because I saw your emotions, I think you have very good composure. Anyway, in the same year, in 1974, Wilson proposed what we now call the lattice QCD, so he put the QCD on the lattice. And then he invented, and then he developed a very beautiful technique to show from this putting QCD on the lattice that, actually, the quark can be confined through the strings.

So the quarks in QCD can be confined through the strings. And that essentially revived the idea maybe the QCD can be a string theory, because the quarks are confined through the strings. And this all happened in 1974.

So then I mentioned the same, in the late '80s and the early '90s, people were looking at these so-called matrix models, the matrix integrals, et cetera. Then they showed they related to lower dimensional string theory. But nobody-- yeah, they showed this related to some kind of lower dimensional string theory. And then in 1993 and 1994, then 't Hooft had this crazy idea of this holographic principle. And he said maybe, things about the quantum gravity can be described by things living on the boundary.

And again, it's a crazy idea. Very few people paid attention to it. But the only person who picked it up is Leonard Susskind. And then he tried to come up with some sort of experiments to show that that idea is not so crazy. Actually, Susskind wrote a very sexy name for his paper. It's called "The World As a Hologram." And so that paper received some attention, but still, still, people did not know what to make of it.

And then in 1995, Polchinski discovers so-called D-branes. And then we go to 1997. So in 1997, first in June, so as I said, that QCD may be some kind of string theory. This idea is a long idea, starting from the 't Hooft and large N expansion, and also from the Wilson's picture of confining strings from the lattice QCD, etc.

But it's just a very hard problem. If from QCD, how can you come up with a string theory? It's just very hard. Very few people are working on it.

So in 1997, in June, Polyakov finally, he said, had a breakthrough. He said that this consistent [? of ?] string theory give you one extra dimension, you should consider a five-dimensional string theory rather than a four-dimensional string theory. And then he gave up some arguments, anyway. And he almost always actually write down this metric And maybe he already wrote down this metric, I don't remember. Anyway, he was very close to that.

But then in November, then Maldacena came up with this idea of CFT. And then he provided [? explicit ?] examples of certain large N gauge theories, which is scale invariant and some string theory in certain anti-de Sitter spacetime. And as I said, through the understanding of these D-branes.

But even Maldacena's paper, he did not-- he was still thinking from the picture of large N gauge theory corresponding to some string theory. He did not make the connection to the holographic principle. He did not make a connection to the holographic principle.

But very soon, in February 1998, Witten wrote the paper, and he made the connection. He said, ah, this is precisely the holographic principle. And this example, he said, ah, this example is precisely the holographic principle Susskind and 't Hooft was talking about.

So that's a brief history of how people actually reached this point. So the next stage, what we are going to do is to try to derive [INAUDIBLE]. So now we can-- as I said, we have two options. We can just start from here, assuming there is CFT [? that's ?] equivalent to some string theory. And then we can see how we can develop this further. And this is one option we can take.

And our other option is to really see how this relation actually arises from string theory. And many people voted for the second option, which in my [? email ?] is option one. So you want to see how this is actually deduced from string theory. So now we will do that, OK? But I should warn you, there will be some technicality you have to tolerate. You wanted to see how this is derived, OK?

So we do a lot of [? 20 ?] minutes today? Without break? Good. OK. Yeah, next time, I will remember to break.

OK. So now we are going to derive this. So first just as a preparation, I need to tell you a little bit more about string theory. In particular, the spectrum of closed strings, closed and open strings. And so this is where the gravity-- and from a closed string you will see the gravity, and from the open string, you will see the gauge theory. OK. We will see gravity and gauge theory.

So these are the first things we will do. So the second thing we will do-- so the second thing we will do is to understand the physics of D-branes. So D-brane is some object in string theory. And it turned out to play a very, very special role, to connect the gravity and the string theory. OK. Connect the gravity and the string theory.

Because this is the connection between the gravity and the string theory. And in string theory, this [? object will ?] deeply and precisely play this role, which connects the gravity and the string theory. So that's why you can deduce such a relation.

OK. Yeah, so this is the two things we will do before we can derive this. So this is, say, the rough plan we will do before we can derive this gravity. So first let's tell you a little bit more about string theory.

So at beginning, just say some more general setup of string theory. So let's consider a string moving in a spacetime, which I denote by M, say, with the metric ds squared equal to g mu mu. And this can depend on X, dX mu, dX mu. OK? So you can imagine some general curved spacetime. Say mu and nu will go from 0 to 1, to D minus 1. So D is the total number of space dimensions for this M.

So the motion of the string, as we said quite a few times now, is the embedding of the worldsheet to the spacetime. So this is in the form of X mu sigma tau. OK, you parameterize the worldsheet by two coordinates. So I will also write it as X mu sigma a. And the sigma a is equal to sigma 0, and the sigma 1 is equal to tau sigma, OK? And we will use this notation.

So now imagine a surface embedded in some spacetime. And this is the embedding equation. Because if you know those functions, then you know precisely how the surface are embedded, OK?

And because the original spacetime have a metric, then this induced metric on the worldsheet. And this induced metric is very easy to write down. You just plug in this function into here. And when you take the derivative, you only worry that sigma and tau, because then that means you're restricted on the surface, when your only [? value is ?] sigma and tau.

And then you can plug this into there. So you can get the metric, then can be written in this form. Here's sigma a and this sigma b. OK? So remember, sigma a and sigma b just tau and sigma.

And this hab is just equal to g mu mu, X, partial a, X mu, partial b, X nu. OK? So this is trivial to see. Just plug this into there, to the variation with sigma and tau, you just get that, and it's that. OK? Is it clear?

So this Nambu-Goto action is the tension-- tension we always write this 1 over 2 pi alpha prime-- dA. So alpha prime is the [INAUDIBLE] dimensions square. So we often also write alpha prime as ls square. So alpha prime, just a parameter, too. Parameterize to [? load ?] the tension of the string.

So this area, of course, you can just write it as d squared sigma. So again, you use the notation d squared sigma just d sigma d tau. d squared sigma minus delta h. OK. So this is just the area, because this is the induced metric on the worldsheet. Then you take the determinant, and that give you the area. So this is the standard geometric formula.

So now let me call this equation 1. So I have a [? lot ?] equation 1 before, but this is a new chapter. OK. So this is the explicit form of this Nambu-Goto action. But this action is a little bit awkward, because involving the square root. A square root, it's considered to be not a good thing in physics. Because when you write down action, because it's a non-polynomial. We typically like polynomial things. Because the only integral we can do is a Gaussian integral, and the Gaussian is polynomial.

So this is inconvenient, so one can rewrite it a little bit. So you write down the answer. So we can rewrite it in the polynomial form. And this polynomial form is corresponding-- it's called the Polyakov action, so I call it SP, even though Polyakov had nothing to do with it. And this action can be written in the following form. And let me write down the answer. Then I will show the equivalent.

AUDIENCE: Wasn't it invented by Leonard Susskind?

HONG LIU: No, it's not Leonard Susskind.

[INTERPOSING VOICES]

AUDIENCE: Why is it called Polyakov--

HONG LIU: Polyakov-- yeah, actually Polyakov had something to do with it. Polyakov used it mostly [INAUDIBLE] first. OK, so you can rewrite it as that, in this form. And the gamma ab is a new variable introduced. It's a Lagrangian multiplier. OK.

So let me point out a few things. So this structure is precisely just this hab. So that's if you look at this structure, so this structure is precisely what I called hab.

So now the claim is adding [INAUDIBLE] to original variable with just X. Now I introduce a new variable, gamma. And gamma is like a Lagrangian multiplier, because there's no connected term for gamma. So if I eliminate gamma, then I will recover this. OK, so this is the claim. So now let me show that.

This is very easy to see. Because if you just do the variation of gamma, do the variation of gamma ab. OK. So whenever I wrote in this is in [? upstairs, ?] it always means the inverse. OK, this is the standard notation for the metric.

So if you look at the equation of motion, [INAUDIBLE] by variation of this gamma ab, then what you'll find is that the gamma ab-- just do the variation of that action. You find the equation of motion for gamma ab is given by the following. So hab, just that guy. And the lambda is arbitrary constant, or lambda is arbitrary function.

So this I'm sure you can do. You just do the variation. You find that equation. So now we can just verify this actually works. When you substitute this into here, OK, into here.

So this gamma ab, when you take the inverse, then [? cause ?] one into the inverse, hab, inverse hab contracted with this hab just give you 2. And that 2-- did I put that 2 in the right place? That gave you 2. And that have a 2 on-- yeah, I'm confused about 2 now. Oh, no, no, it's fine.

Anyway, so this contracted with that, so gamma ab contracted with hab give you 2 divided by lambda, times 2. OK? Because you just invert this guy and invert the lambda and 2. And then square root of minus gamma give me 1/2 lambda, square root minus h. OK?

So sometimes I also approximate. I will not write this determinant explicitly. When I write [? less h, ?] it means the determinant of h. And the minus gamma, determinant of gamma. OK?

So you multiply these two together, so these two cancel. And this two, multiply this 4 pi alpha prime, and then get back that, OK? So they're equivalent. Clear? So this gives you [? SNG. ?]

So now the key-- so now if you look at this form, this really have a polynomial form for X, OK? So now let me call this equation 2. So equation 2, if you look at that expression, just has the form-- so this is just like a two-dimensional field theory-- has the form of a two-dimensional scalar field theory in the curved spacetime. Of course, the curved spacetime is just our worldsheet sigma with metric gamma ab, OK?

So this is just like-- but the key here, so sometimes 2 is called the nonlinear sigma model, just traditionally, a theory of the form that equation 2 is called the nonlinear sigma model. Nonlinear because typically this metric can depend on X, and so dependence on X is nonlinear. So it's called nonlinear sigma model.

But I would say it's both gamma ab and X are dynamical. Are dynamical variables. So that means when you do the path integral, so in the path integral quantization, you need to integrate over all possible gamma ab and all possible X mu. Not only integrate over all possible X mu, but also integrate all possible gamma ab with this action.

OK. So this is a two-dimensional [? world ?] with some scalar field. And you integrate over all possible metric, so over all possible intrinsic metric in that [? world. ?] So this can also be considered as 2D gravity, two-dimensional gravity, coupled to D scalar fields.

So now we see that when you rewrite anything in this polynomial form, in this Polyakov form, the problem of quantizing the string become the problem of quantizing two-dimensional gravity coupled to D scalar fields.

OK. So this may look very scary, but it turns out actually two-dimensional gravity is very simple. So it's actually not scary. So in the end, for many situations, this just reduced to, say, a quantizing scalar field with a little bit of subtleties. So yeah, let's stop here.