HONG LIU: So last time, we introduced this threshold metric for the black hole. And then this rs gives the horizon size of the black hole. And then, there are two very important geometric quantities for the black hole. One is the so-called area of the horizon, which is the area of this s2 at the horizon. And that the other is what is called the surface gravity, which is essentially given by 1/2 of the derivative of this function, f, evaluated at the horizon. And if you value it on this, then this gives you 1 over 2 rs. OK. So those quantities, we will see them very often.

So now we talk a little bit more about the causal structure of the black hole. So before I start, do you have any questions?

OK. So to understand the structure, the space-time structure of a black hole, a little bit better, let us look at the geometry view of the horizon. Because horizon is clearly one of the most important part of the black hole geometry. So let's consider the region near the horizon. So this is for r greater than but close to the Schwarzschild radius.

So, for this purpose, you can expand this function, f, so near the horizon you can say rs plus f prime, et cetera. And this sum is equal to 0, so we essentially, when you expanded this function f, so that's essentially what you have. OK. Then here, then they have all the terms, but they can be neglected if we are close to the horizon.

So let's put this insight into here. But to write this metric in a slightly more transparent form, it's convenient to consider the proper distance. Also consider the proper distance. Say suppose you are a stationary observer sitting at some, say, some value of r, then you're asking what is the proper distance from the horizon? And that can be just read from, by integrating this distance. So essentially, d rho, so rho, the proper distance, should given just by dr divided by square root of f. OK? You can read it from here.

AUDIENCE: Can you-- can you give us a reminder what's proper distance?

HONG LIU: The proper distance is the distance of a local observer. Just from local observer the [INAUDIBLE] looks like a flat space-time, and then that's the distance measured by that observer. Yeah. Similarly, which is the local proper time, et cetera. So then, by using this, then we find here the horizon, then we can write this as, say, dr plus higher order corrections, so now if you integrate this rho, and the setting rho equal to 0 to be the horizon, then we find rho equal to-- yeah, I just integrate that equation, which can be easily integrated, under the integration constant is chosen, so rather rho is equal to 0 add to the horizon, then you find the proper distance from the horizon is proportional to the square root r minus rs of this. Yeah.

So now we could rewrite to the f itself, in terms of this rho, OK? So if we rewrite the f in terms of rho itself, so f is just f prime, r minus rs. And a square root r minus rs equal to rho times this factor. So we said this be algebra, you can see plus higher order corrections. OK? And this oh, so now you recognize this is essentially just the surface gravity. So near the horizon, this function f can be written as kappa square, rho square, then plus higher order corrections.

So now we can rewrite to the horizon. Now we can rewrite this metric in terms of this rho. The ds squared. Now equal to [INAUDIBLE] Kappa square, rho square, dt square, plus the rho square. So you're thinking about definition, this term just becomes 0 square. And f I replace by this kappa square rho square. And to the leading order, I just replace this r square by rs, because we are close to the horizon.

So now I can slightly rewrite this. So I introduce a variable. So let me introduce a variable, eta, equal to kappa times t. So essentially just t divided by 2 rs. So with that range of eta, then this becomes minus rho square to eta square, plus d rho square, then plus rs square d omega 2 square. So this is just a s2. This is a two-dimensional sphere with a radius given by the horizon sides.

And then these, some of you may already recognize. This is, in fact, just a Minkowski spacetime. This is a 1 plus 1 dimensional Minkowski spacetime. OK? So this is actually a 1 plus 1 dimension Minkowski's time retain in the special coordinate. So, in the so-called Rindler form. The so-called Rindler form. OK? So we see then either the horizon, the spacetime geometry have the structure of a 1 plus 1 dimensional Minkowski spacetime, times a two-dimensional sphere.

So now let me elaborate this a little bit, on why this is Minkowski's space. OK? So do you have any questions so far? OK. Let's just start. Let's consider 1 plus 1 dimensional Minkowski spacetime. OK, now you're going to see 1 plus 1 to here, done. I will write it in the following form. I call my T-- So to distinguish with the t and there-- so let me call it capital T, and capital

X. OK? So this is our familiar 1 plus 1 dimension Minkowski's time.

So now I will introduce a new variable. So make a coordinate change. OK? So I just plug this into there. So let me call this m2. So you just plug in them there, and you immediately see that ds square m2 is just precisely we see there. OK? So this metric, so this part of the metric, you see, in fact, flat. But this coordinate, rho eta actually does not cover the full Minkowski space, because, just by definition, this coordinate satisfy x square minus t square equal to rho square. Because this is [INAUDIBLE].

So this, by definition, satisfy x square minus t square actually greater than equal to 0. And also, just by definition, because we will take rho to be positive, and by definition, the x is positive. OK, we will take rho positive. Greater/equal to 0. So let's take rho greater/equal to 0. And which is what we are having here, because this rho is related to the proper distance from the horizon is, by definition, a non-elective quantity. So here we also want to restrict to rho greater/equal to 0.

And so it's easy to draw what part of Minkowski spacetime this is. Let's plot it-- T, X-- so we have the light cone. So this occupies the region which x-- so x can be positive, so have to be on the right side. But also, we have x, the absolute value of x must greater than absolute value of t, because this thing is greater than 0. So that means it must be cover this region. And this is, let's say, this is [INAUDIBLE] x equal to t. And this is T minus equal to x. So these are the two light cone, and so they can only occupied that region.

You can also show-- let me erase those things here, so it's only in this quadrant. So now let me erase these, so that I can draw further lines. Not to make things too messy. So you can also just invert this relation. So if we invert this relation, then you see that the rho square equal to x square minus t square under the tangent times eta equal to t divided by x. So that means that the constant rho should correspond into constant x square minus t square, and that's just hyperbola. So this is a constant rho. Say this is rho equal to constant. OK?

And from here, the constant eta will corresponding to constant slope between t and x, so the constant eta will be corresponding to the straight line, like this. OK? So, essentially, we see that this metric covers this part of the Minkowski 1 plus 1 dimensional them Minkowski's time, but 48 the Minkowski's time using coordinate like this. OK?

So now, on this light cone, so there are two special lines here. One is x equal to t, which is essentially the boundary of this region, so here is when rho equal to 0. So now you look at is formula, where rho equal to 0, when rho close to 0-- oh, maybe we should do it in the limit. So when rho approach 0, in order for x and t finite-- so on this [INAUDIBLE], of course, x and t are finite. Then the eta has to go to infinity. So that means the light cone [INAUDIBLE] rho equal to 0, and the eta goes to plus infinity. OK? And the [INAUDIBLE] rho exponential eta finite. And similarity, t equal to minus x, this line, then goes 1 into rho goes to 0, but now eta goes to minus infinity. And when eta goes minus infinity, then the minus sign here will dominate the t and the rho exponential minus eta finite. OK?

And so there's a special point here that goes 1 into x equal to t equal to 0. So this goes 1 into rho equal to zero, and any finite eta, say, will correspond into that point. OK?

So we see that this is precisely what describes the black hole horizon. OK? And if back here you've got rho equal to 0, precisely lie on the black hole horizon, so from here, we conclude, so the black hole horizon is corresponding to x equal to plus minus t. OK? Here you also see, so this rho and the eta is essentially simple transformation of what we have here, r and the t. So we also see, just from this discussion-- so we see that the black hole horizon just is mapped to the light cone in the 1 plus 1 dimensional Minkowski's time, and that the region outside the black hole just mapped to that quadrant of the Minkowski spacetime.

So we conclude that the little black hole geometry, near horizon black hole geometry. So this is called Rindler space. So we see that the near horizon black hole geometry is just Rindler times s2. OK? In particular, it's precisely have this causal structure. You have a light cone which comes 1 into the horizon. So that also shows that the horizon is a long surface, because the horizon comes 1 into the light cone. And so we can just directly translate that picture from rho into t and r. OK? So this is just r. You go to rs. So eta goes to-- So here, is that the rho equal to 0, of the eta goes to plus infinity. So here is the rho goes to 0, and eta goes to minus infinity.

So that means, for the black hole, this will cause 1 into i equal to i s, and the t goes to positive infinity. And t and eta just differ by this constant factor, and so essentially the same thing. And here, rs t equal to minus infinity.

And then similarly, near the horizon, the constant r surface will be like this, just like a hyperbola in the-- and then the constant t surface will also like this. OK? And then, of course, the full spacetime would be times s2, so you should imagine each point on this plot, which only plot r on the t plane, so each point there this s2 with this radius over i rs. OK? Any questions on this? Yes?

AUDIENCE: What's the meaning of the arrow on r equals constant hyperbola?

HONG LIU: Sorry?

AUDIENCE: What's the meaning of the arrow on the r equals constant hyperbola?

HONG LIU: Ah. That's just come from here. So from here, once we map to these coordinates, we see that the rho constant because 1 to hyperbola like that. And the rho is essentially just related to r by a simple transformation. So constant the rho surface is also the constant the r surface, because they are related to a similar transformation. So the constant r surface must also be a hyperbola like that.

AUDIENCE: But why is there an arrow pointing to that?

HONG LIU: Oh, the arrow just to show that the time goes up. Yeah, Here you can also show arrow, which is the direction which the eta increases. Yeah, so this is the direction which time goes up. And the time goes up, and, yeah. Other questions?

OK. Now let me make some further remarks. So some of them just repeat what I just said, just so that we have it here. So observer at r equal to constant, but r say slightly outside the horizon for black hole, black hole geometry corresponds to an observer with rho equal to constant in Rindler patch of m2, OK, of two-dimensional Minkowski spacetime.

And then this is observer which follows a hyperbolic trajectory. OK? So it goes like that. So this is not the standard observer. This is a hyperbolic observer. Yeah? So such observer which has a constant r, they're not the standard initial Minkowski observer. They travel in hyperbola.

So now you can check easily, just by working with this flat space, because we know everything about flat space. You can easily work out. So you can easily check yourself. It's an elementary calculation. Take you a couple minutes. Such observer has a constant proper acceleration, which is given by a equal to 1 over rho. OK?

So you can also translate into black hole language, when we converted this rho into rs. So this becomes 1/2 rs. Yeah. Then they make us just including the whole thing.

So you can see from this formula, so this is a local proper acceleration of observer, as we discussed yesterday. If you want to be-- because a black hole attracts things, so if you wanted to stay at a fixed radius outside the black hole, you need to accelerate. You need to have some acceleration. And this is just that the acceleration close to the black hole.

And, in particular, you see this accelerating goes to infinity when you approach the horizon. OK? When you approach the horizon. So just a side remark-- let me put here, side remark. So this is related once the last remark, the last thing we said yesterday, is that the surface gravity is the acceleration of an observer at the horizon, near the horizon viewed from infinity. OK? So the infinity. So that formula, so that ar, so that formula is the local acceleration. And you can easily convince yourself the acceleration at infinity is related to the local acceleration by a [INAUDIBLE] factor precisely given by this. And if you just take that formula, and multiply it by this, then you find precisely 1/2 f prime rs, which is kappa. OK? Which is kappa.

So this also give you a derivation of this surface gravity, this formula for the surface gravity.

AUDIENCE: I have a question. Why is [INAUDIBLE]?

HONG LIU: Yeah, that's where I ask you to check yourself. Yeah, I said it's a couple minutes calculation. It's flat space. So you should be able to do it. Any other question? Yes?

AUDIENCE: What's the definition of proper acceleration?

HONG LIU: Yeah, it's acceleration in the local rest frame of that observer. Yeah, anything you call the proper is something which is defined in the local rest frame of that particular observer. Yeah.

AUDIENCE: What's the expression for the [INAUDIBLE]?

HONG LIU: Yeah. It's, say the a mu, say it's a vector, say it can be defined a mu equal to u nu plus nu u nu. And so, this u nu is the local trajectory of that observer. So this is the definition. You can see it in the TR book. So that's the local-- so that's a velocity vector of the observer. U mu.

OK. So this is the first thing. The second thing-- Yeah, so if you check in this Waltz book which I wrote last time. I put the reference in the Waltz book, and he can find it. Or he can find it in any GR book, they explain it.

So now, a free-fall observer near the black hole horizon, they just gave you, just goes one into inertial observer in Minkowski spacetime, OK, in M cube. So this is easy because they map to each other. So free-fall observer is observer in the black hole spacetime, which just follow from, say some time like geodesics. And we know that in the time like geodesics in the Minkowski space is just straight line. Just straight line. So for object fall into to the black hole, you just follow a straight line. So this provide the very simple description of an [INAUDIBLE] observer near a black hole.

So this is the second remark. So the third remark is that this Rindler coordinates rho eta, as we see from here, become singular. And the rho equal to 0. Yeah, you see to describe those rho equal to 0 to describe the horizon, you have to take a [INAUDIBLE] limit. There's no longer one-to-one relation between the capital X and T, and the rho and the eta.

So, for this just explains that the Schwarzschild coordinates-- so the same thing happens with the Schwarzschild coordinate at the horizon. In the same way the Schwarzschild coordinates become singular at the horizon. OK? At the same way. But we do understand in Minkowski spacetime, we know that the reason this-- we do know the actual Minkowski spacetime have four patches, rather than just a single patch, rather than just the single quadrant. And so, just by changing the [INAUDIBLE] to a different coordinate, this capital X and capital T, we can actually describe the full Minkowski spacetime, not just the single quadrant. OK?

So let me call this region one. So now let me call this, so this Minkowski spacetime separated by light cone into four regions-- I, II, III, IV. OK. I hope this is clear. There are too many things on the blackboard. But by this quadrant is I, this top quadrant is II, this right quadrant is III, and the bottom is IV. OK?

So even though this Rindler coordinate become singular rho equal to 0, by using Minkowski coordinates, standard Minkowski coordinates, x and t, we can actually extend this region I to the full Minkowski spacetime. OK? To the full m2. To the full Minkowski spacetime.

So now he's becoming mechanical. So now, you must be, because this is essentially what describes the black hole near the horizon. Then in the black hole you must be able to find the set of coordinates to actually extend beyond the horizon. OK? So, similarly, by changing to suitable coordinates, one can-- so this is normally called the Kruskal coordinate-- we will not need to use it, so I will not write it down explicitly. It's just a generalization of this-- going to this capital X and T. So in the black hole geometry, by going to these Kruskal coordinates, the counterpart of this Minkowski coordinates, one can also extend the Schwarzschild geometry to four regions. OK?

Just like this region in Minkowski can be extended to four regions, and this region outside the black hole horizon can be extend to four regions. OK? And we can, in fact, immediately just modeled on here, going from here to the full Minkowski spacetime, we can just immediately extend this, so we can immediately extend the black hole geometry into four regions.

The only difference-- of course, this picture is only precise when you are close to the horizon. So once are you far away from the horizon, so those corrections will become important. OK? But those don't change the basic causal structure of the black hole, which are essentially determined by the horizon structure. So essentially, just add to the-- in terms of the causal structure, this picture just extends to the full black hole spacetime. OK?

And in particular, say if go this way, go to r, go to infinity, and there's a slight subtlety, slight difference, from this picture, it's because black hole eventually becomes singular when r equal to 0. So black hole have a singularity at i equal to 0. And that happens inside the horizon, so this happens in this region. So eventually you will reach i equal to 0. You will reach some singularity, so we will draw it by some wavy lines. And also there's a singularity in the past.

AUDIENCE: Question. Is the Gordon transformation the same as from Rindler to flat?

HONG LIU: It's very similar. Of course, it's not identical, because that Rindler is only valid close to the horizon, and we're away from the horizon things become different. Yeah, but qualitatively, they're very similar. The precise mathematical form, of course, are not the same.

AUDIENCE: And on the horizon, they are the same.

HONG LIU: Yeah, essentially the same. Any other questions.

AUDIENCE: I have a question.

HONG LIU: Yes.

AUDIENCE: Does it make sense, just from a mathematical point of view, to have a piece of Minkowski spacetime on its own? You just, the manifold is only that quadrant?

HONG LIU: Yeah, sure. Sure.

AUDIENCE: Could a similar thing happen in the case of a black hole, that the inside is simply the manifold does not extend into it?

HONG LIU: Sure. Yeah. It can happen. Yeah. But, one thing, so here we are making a basic assumption, is that you can fall into a black hole. And from this picture, we know that falling into a black hole, from Minkowski point of view, means something very trivial. Just cross that light cone.

And so this region must exist. So let me call this I, II, III, IV. So this region must exist. OK?

Yeah, so let me make some further remark. In this three, then it may become clear to you. Say these are the remarks for this diagram. He said when you fall into the black hole, you enter region II. Because time goes up, you enter region II. And clearly, because this is a light cone, so if you draw a light cone inside the horizon, no matter what point you draw the light cone inside the horizon, there's no way you can cross outside across this surface. OK? So clearly, no information or observer in region II, can reach region I. OK?

So we call this a future horizon. We call this a future horizon, because once you go beyond it, you cannot influence here anymore. So we call this a future horizon. OK?

So another comment is that regions III and IV, just like in Minkowski case, can be obtained. from I and II by time reversal. OK? Yeah, actually, not precisely. Only region IV. Only region II. Let me see. Region IV can be obtained from II by time reversal. Yeah, by time reversal.

So for the real back hole, for region III and IV do not exist for real black holes formed from collapse. Formed from collapse. OK? So in some sense, regions III and IV are artifact of we are working with a time reversal in [INAUDIBLE] geometry.

So also, again, if you draw the light cone in the region I, there's no way you can send signals to region IV. OK, again, because of the light cone, it's going that, so there's no way you can cross into the-- so the light radius no way can reach region IV. So observer in region I cannot influence-- oh, this is already finished. Influence IV. Maybe I should go back to here. Cannot influence region events in region IV. So that's why this one is called the past horizon. OK? So this is called the past horizon.

So the final remark r the diagram, is that r equal to 0. Of course should be. So if you go into the r equal to 0 cause light inside to the horizon. And so r equal to 0 is the black hole singularity. So notice that outside the horizon, the constant r surface goes like this, OK. But when you go inside the horizon, as we remark before, the f will change sine, then the rho between t and r will switched. So r actually become time. So that's why r equal to 0 like a constant time surface, OK? And so is like this. And then this just a time reversal of that. So this is called the spacetime singularity. So this is a spacetime singularity, because r become time. Yes?

AUDIENCE: So this is a time reversal in variant geometry, but so is it possible for there to be time reversal, a time reversal in processes that are not time reversal? That you cannot time reverse in a time reversal in variant geometry?

HONG LIU: Of course, in the time reversal in variant geometry, there certainly exists processes which it does not respect. Time reversal in variants. That can happen. But for each such process, there's a mirror process.

AUDIENCE: So in the case of crossing from I to III, I guess you can cross from I to II, but not from II to I, but you can cross from IV to I?

HONG LIU: Yeah, you can cross IV to I.

AUDIENCE: So that's like a mirror process?

HONG LIU: Yeah, That's right.

AUDIENCE: [INAUDIBLE] what does it really mean about the inside the horizon the space and time [INAUDIBLE]? I mean, mathematically it's [INAUDIBLE]?

HONG LIU: Physically, if you are a local observer you will not feel anything. From a local observer point of view, when you cross the horizon-- So this is just related to our choice of coordinates. Just imagine from here, we'll cross from here. Of course, nothing really happens. Yeah. Yeah. Here, the reason-- yeah, just the t, we just related to the rho of t and r switch. T and r, they just coordinated. They're not-- Yeah, it's just how you describe the system. You can prefer not to use t and r. Then cross the horizon, nothing really happens.

AUDIENCE: [INAUDIBLE] euclidean formula, you cannot cross the horizon.

HONG LIU: Yeah, and we will talk about euclidean later. Any other questions? Yes?

AUDIENCE: Why do you specify that it's real black holes from collapse?

HONG LIU: Maybe I should just equal black holes, not imaginary black holes. Yeah, just say black holes. Yeah, real life black holes.

[LAUGHTER]

Yeah, then I was getting neglected [INAUDIBLE]. Any other questions?

AUDIENCE: [INAUDIBLE] space lights.

HONG LIU: Sir, you have something.

AUDIENCE: And don't they take infinite time to form? Don't they take infinite time to form?

HONG LIU: The black hole? No, they take-- gravitational collapse always take finite, always take finite proper time. Say, from the observer in infinity, and you will have the phenomenon because of the infinite redshift. And then, seems like they will never pass through the horizon. But eventually they will. Eventually you cannot see them, because the light ray will become dimmer and dimmer, et cetera. Yeah. Anyway, so this is a standard question in GR of what happens before the black hole collapse. Yeah, they do collapse. Does this answer your questions? No? OK.

[LAUGHTER]

Yeah. I cannot do the whole GR course here. Yeah.

AUDIENCE: I just wanted to ask-- so how long does it take for the observer's frame? The time it takes from the horizon to hits the r equals to 0 singularity.

HONG LIU: Well, then you just calculate it.

AUDIENCE: Is it finite?

HONG LIU: Yeah, it is finite.

AUDIENCE: Then how does it feel? I mean--

[LAUGHTER]

[INAUDIBLE] suddenly stop when you hit the singularity?

HONG LIU: Yeah. Before you hit the singularity, you'll already be killed.

[LAUGHTER]

So the curvature will become very big. The curvature will be very big. And, yeah.

AUDIENCE: [INAUDIBLE] quantum fluctuation spacetime.

HONG LIU: Yeah. So what we believe is that this singularity-- so the curvature will become infinite here. So before you reach here, you're already, say, be killed by the tidal force, et cetera. But we do believe that the curvature, this curvature singularity, must be just a mathematical artifact of using the classical GR. Say, if you are able to use quantum gravity, then there should not be a singularity. But there are even debate whether they're even reaching behind the horizon. Maybe you can never cross the horizon. Maybe you already get killed at the horizon.

[LAUGHTER]

That's a lot of possibilities. That's a lot of possibilities. Yeah. So we don't really know. So here, we are using the assumption that the things, so there's nothing wrong, at least from the classical metric, nothing wrong on the horizon, then you can cross it. But there have been debate whether you can really cross it or not.

AUDIENCE: But you'll get killed anyway.

[LAUGHTER]

HONG LIU: But nobody knows. Yeah, even if you get killed, no other people will know.

[LAUGHTER]

AUDIENCE: Are some people working on experiments? [INAUDIBLE] we have an academia response. So we add a prohibition to a system to see how it responses. For a black hole, if [INAUDIBLE] what kind of experiments?

HONG LIU: Yeah. I think in the second half of the class, we will see such examples. We will. Yeah. Some process, some scattering process reached the black hole. That can be considered as some kind of linear response. Yeah. Other questions? OK, good. So I'm running a little bit short of time. Maybe I should-- So does everybody know what's a Penrose diagram?

AUDIENCE: No.

HONG LIU: So not? Half? Maybe half? Raise your hand, people who know Penrose diagram? OK. Yeah, let me maybe skip it now--

[LAUGHTER]

Because we will not immediately use it. So let me just skip it, just because I want to reach the next topic today. It's a little bit more interesting than the Penrose diagram.

OK. So these are all essentially the classical physics of a black hole. So now we can go to a little bit quantum. So Hawking, in 1974, made the great discovery that actually, you don't need to treat the black hole as quantum. If you just treat the matter field, treated the matter, whatever matter outside the black hole-- so photon, electron, neutrino-- treated those things as quantum in the back hole geometry, then actually, then black hole does not appear as completely black. Actually, black hole radiates like a black body. So actually, black hole have a well-defined temperature. OK?

So now, we will try to derive this black hole temperature. There are many ways to derive this. The most authentic away is essentially Hawking's original derivation. To really show that if you have a gravitational collapse, and then long after the black hole has formed, actually there is a still [INAUDIBLE] radiation coming out of the black hole, from the perspective of the observer at infinity. And so you can deduce that actually black hole is like a black body with a finite temperature.

And you can show that the radiation is precisely [INAUDIBLE] in the semi-classical approximation. Say treating the geometry as classical, but treat the quantum field as quantum. And then you can show that the spectrum is exactly [INAUDIBLE]. But we will not go through such a derivation, because it takes a little bit of time. And also, we will not lead that particular technical tour de force. Yeah, it's a technical tour de force, but we will not need it.

So I will give you a different derivation. And this derivation is very simple, but it's a little bit sleek. It hides a lot of things. And so I will just try to give you a derivation first, and then we will talk about the funny things which are hidden by this derivation. OK?

So first, let me remind you, that in the quantum field theory, if we want to describe a system at finite temperature, we go to euclidean, one way to do it is to go to euclidean signature. We take t equal to minus i tau. And then you periodically identify tau to have a radius, to have a period identify tau to have lens h bar times beta, and the beta is 1 over temperature. OK? So there's a key thing, this h bar here. And we will always take the Boltzmann constant to be 1.

So conversely, if you have a euclidean theory which have a periodic time, then when you analytic continue back to the Lorentzian signature, then that system should be at a finite temperature. OK? Should be at a final temperature.

So now let's try to analytic continue. So now the logic is the following. So now the logic is that, let's now try to analytic continue the black hole spacetime to euclidean signature. And then I will show that the euclidean time of a black hole is forced to be periodic. With some specific periods. And so this tells you, by consistency, the black hole must have a finite temperature. OK? So that's the logic.

So now let's do a [INAUDIBLE]. And let me keep this here. So, for black hole, so let's take t equals to minus i tau. OK? Then the euclidean black hole geometry. So now, again, I go to the horizon. I go to the horizon, and I just take this thing. So this is a Lorentzian metric near the horizon. So now I now go to the horizon. Let's go to the horizon. Then we find that this euclidean metric, then just you take this t equal to minus i tau, then you find rho square kappa square tau square plus t rho square plus r square d omega 2 square. OK?

So again, in analog with here, I introduce a euclidean version of this eta, so I introduce a euclidean version of this eta, which is [INAUDIBLE] theta. So I introduce, let me define, introduce, theta equal to kappa tau. OK? So you do that-- so now I can erase here-- then you find this becomes rho square d theta square plus d rho square plus r s square d omega 2 square. So now, this we recognize without any explanation. This is just euclidean flat space. Euclidean flat two-dimensional space in polar coordinates. OK?

But with one difference, is that, in the standard euclidean coordinate, the theta is periodic in 2 pi, but here, for the black hole case, this tau, in principal, is uncompact. OK? But this metric, as a singularity, has a conical singularity at the rho equal to 0 for any theta not equal to 2 pi. OK? For any theta not equal to 2 pi, there's a conical singularity. So this is easy to understand. If you have things which are now equal to 2 pi, you can fold it, then become a cone, and then tip of the cone is singular. For any theta whose periods not equal to 2 pi. I think you understand the sentence.

A singular-- yeah, let me write proper English. It has a conical singularity unless theta is periodic in 2 pi. This is proper English.

So, now the little horizon metric will become like this. And then we see that this metric is actually singular at rho equal to 0. If theta is not periodic in 2 pi. But as we said, in the Lorentzian signature, the horizon is a completely smooth place. Nothing should happen there. In particular, from this Rindler picture, you can just pass through the horizon. This just flat space. It's nothing really happening there.

So since to the horizon non-singular in Lorentzian picture, in Lorentz signature, it should not be singular in euclidean. OK? So we conclude that this tau that must be periodic, so that this redefine, this theta, should have period in 2 pi. OK? So that means tau must be periodic, with a period given by 2 pi divided by kappa. OK? So that theta has period in 2 pi. Yes?

AUDIENCE: Isn't that metric slightly different from the original one, because here, rho is forced to be greater than 0?

HONG LIU: No, rho equal to 0 is the horizon, right?

AUDIENCE: But negative rhos are not covered in this subject. But negative rhos are excluded.

HONG LIU: Yeah. In the euclidean spacetime, negative rho is also not allowed. In euclidean flat space. It's the same thing here. Rho is a radial coordinate.

AUDIENCE: In Minkowskian time, we had the region I and II, like inside and out.

HONG LIU: Right. But once you write in this coordinate, once your write in this form, euclidean form, and then this is just really a flat space. And rho is really a radial coordinate. You cannot extend rho to negative value.

So recall that this thing should be identified as h beta. OK? So now, if you consider, say, a quantum field theory in this black hole spacetime, then that quantum field theory must have a periodic imaginary time. And then this should be identified with this h bar beta. So this tells us-- let me just write-- so let me say recall t is the proper time. So this tau is the euclidean version of this t, but each observer a different r, they have a different proper time, and t is only the proper time for the observer at infinity. OK? [INAUDIBLE] the proper time for observers at r equal to infinity. So we deduce from here that observers at r equal to infinity must feel a finite temperature given by.

So, by identify this thing to be h bar beta and beta is 1 over t, so that means t is equal to 1 over beta is equal to h bar kappa divided by 2 pi. OK? So this kappa is 1 over 2 s. So this is 2 pi r s, and this is h bar 8 pi GNm.

So we conclude if you want to put a quantum field theory in this black hole spacetime, then this quantum field theory is automatically at the finite temperature, with this temperature. OK? With this temperature. So to emphasize, this is a temperature to the observer at r equal to infinity. So you can easily work out the local temperature for observer at some r. Then you just work out the redshift factor, since the t local, so as we discussed yesterday, is one half r d t of the infinity.

So that tells you, and the temperature should be conjugated with the time, [INAUDIBLE] time, so that means that the t local r should be, that the t at infinity minus one half r. ? OK Then t for infinity is this one. So, yeah, just call it t. So this is just h bar kappa divided by 2 pi f minus 1/2 r. And in particular, this will goes to infinity, as r goes to r s. Because this f goes to 0 when you approach horizon, because it lacked in power, so this goes to infinity.

So the local temperature at the horizon is very, very hot. So at infinity, you have this temperature. But when you get closer to, closer to horizon, for any station observer, you feel hotter and hotter, and then become infinite temperature of the horizon. OK? So black hole is actually a very hot place, if you want to get close to it. Yes?

AUDIENCE: But didn't-- But this metric is still the one near the horizon.

HONG LIU: Yeah.

AUDIENCE: So how can we make sure that that's indeed the temperature at infinity? I mean, it seems that there's other corrections that we have to do. [INAUDIBLE].

HONG LIU: Yeah, this is a good question, The only thing matters, the only thing matters, is what time you use to periodically identify. So the time we identify is the time which is used at infinity.

AUDIENCE: I see.

HONG LIU: Yeah, because we analytic continue this time, and it's this tau get identified. And this tau is the tau used by the observer at infinity.

AUDIENCE: I see.

HONG LIU: Yeah. So observer in other positions, they use different tau obtained by analytic continuation of this guy. So that's why, then, you will see a different temperature.

AUDIENCE: Well, no. I'm asking, this is not the actual-- we're using a [INAUDIBLE] metric, or [INAUDIBLE] metric, right?

HONG LIU: We're using?

AUDIENCE: We're not using the exact metric, are we?

HONG LIU: Oh, we are using the exact metric. I mean, this doesn't-- so I know what you're asking. This tau is the exact tau for the-- t. We have not changed the t.

AUDIENCE: Oh, OK. I see what you're saying. Yes.

HONG LIU: We just look at the behavioral of the metric near the horizon. But t is still the t used in the regional metric, and which is the t used by the observer at infinity. Other questions?

OK. Good. So you may say, r, OK, fine, the black hole will have a finite temperature. But now, if we use this relation with a Rindler spacetime, then we immediately conclude, in Minkowski spacetime, some observer must also see a finite temperature. OK? So let's just-- So let's do, for the Rindler space-- OK? So let's write is as so this is a Rindler space Lorentzian metric. Now we take, again, go to the euclidean signature. Let me call eta equal to [INAUDIBLE] i theta. And then I get the euclidean metric, which just-- yeah, of course it's just exactly the same thing that we did for the black hole, because they are the same thing. Near the horizon is same thing. So that means the theta must be identified with 2 pi. OK? But now, you say, observer who use this time, eta, must feel at the finite temperature. OK? So the time, which time you use, is crucial. And this eta is not the standard Minkowski spacetime. And this is the time which-- yeah, I forgot-- which is the time. So for the Rindler space, is the time which goes like this. Yeah, it's the time which goes like this. This is not the standard Minkowski time.

So for observer who are using that time, then they should feel to be at a finite temperature. Of course, the observer who used that time is precisely the observer who are at the constant rho. So if you have a constant rho, then d rho equal to 0, then that's the time you use. OK? So, but with a slight correction-- so for observer. So this is normally called a Rindler observer at constant rho, at rho equal to constant, then it's local proper time as given by this guy, then t, d t local is equal to just rho d eta. OK? Then that means the corresponding euclidean time is related to this theta, also by this factor of rho, is rho d theta. And that means that this tau local will be periodic in 2 pi times rho. OK?

And again, you identify this with h bar beta. So that's tell you for Rindler observer, at location rho, you will see a temperature which is h bar divided by 2 pi rho. And, as we said before, 1 over rho-- I just erased it-- is essentially acceleration. So this is just h a divided by 2 pi, and a is the acceleration. So we call a equals to 1 over rho. So the acceleration is equal to one over rho. So this tells you, once you write it in this form, once you write in this form, you can forget about Rindler space, because the only physics is now using the acceleration.

OK? Now you can say any Minkowski observer with a non-zero acceleration, will feel to be at the finite temperature. OK? Will feel at the finite temperature. And the finite temperature is proportional to its acceleration and the h bar. Yes?

AUDIENCE: And so an observer free-falling into a black hole will not--

HONG LIU: Yeah, will not feel temperature.

AUDIENCE: So 0 temperature? So even like when you're approaching the horizon and you're close to the horizon, usually you would have an infinite temperature, but here you're going to have a 0 temperature?

HONG LIU: Yeah. That's right. Yeah, so here, and besides, this must be observer is some constant value of r. Yeah, must be a stationary observer outside the black hole.

Yeah, so let me just write down this final sentence. You said, in Minkowski spacetime an accelerated observer will feel the temperature proportional to acceleration. So this was first discovered by Unruh. So this is called Unruh temperature. So this was discovered by Unruh, I think also around 1974. Maybe 1976. Around the time of Hawking's discovery of Hawking radiation. So Unruh, actually, was very close to derive the Hawking radiation himself. Yeah, there were several people who were very close to derive the Hawking radiation, but only Hawking, in the end, did it. But then, Unruh realized, actually Hawking radiation is actually the same as this phenomena in flat space. And so, now this is called Unruh temperature.

OK. So as I said earlier, that this derivation is very simple and powerful, and we will use it again and again in the future. But it's quite sleek. It does not tell you where this radiation, where this temperature comes from. And if you have a black body radiation, where does the radiation come from, et cetera. And so next time, we will try to explain, physically, where does this temperature come from? Yeah. OK. Let's stop here.