8.821/8.871 Holographic duality

MIT OpenCourseWare Lecture Notes

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Lecture 3

Rindler spacetime and causal structure

To understand the spacetime structure of a black hole, let us consider the region near (but outside) the horizon.

Introducing the proper distance ρ from the horizon:

$$d\rho = \frac{dr}{\sqrt{f}} \stackrel{r\to r_s}{=} \frac{dr}{\sqrt{f'(r_s)(r-r_s)+\cdots}}$$

Solving the above equation we have

$$\rho = \frac{2}{\sqrt{f'(r_s)}}\sqrt{r-r_s}+\cdots$$

Then we can express f as a function of ρ

$$f(r) = f'(r_s)(r-r_s)+\cdots = (\frac{1}{2}f'(r_s))^2\rho^2+\cdots = K^2\rho^2+\cdots$$

where K is the surface gravity defined in Lecture 2.

Near the horizon, we have

$$ds^2 = -K^2\rho^2dt^2 + d\rho^2 + r_s^2 d\Omega_2^2 = -\rho^2 d\eta^2 + d\rho^2 + r_s^2 d\Omega_2^2$$

here we define $\eta = Kt = \frac{t}{2r_s}$. The first two terms in the above expression is called (1+1)d Minkowski metric in a Rindler form.

To see this, consider $\mathcal{M}_2$ (2d Minkowski spacetime):

$$ds^2_{\mathcal{M}_2} = -dT^2 + dX^2$$

let $X = \rho\cosh\eta$, $T = \rho\sinh\eta$, then

$$ds^2_{\mathcal{M}_2} = -\rho^2 d\eta^2 + d\rho^2$$

But since $X^2 - T^2 = \rho^2 \geqslant 0$, (ρ,η) coordinates only covers a part of $\mathcal{M}_2$. And $\rho \geqslant 0$ sector corresponds to $X \geqslant 0$ *i.e.* region I as shown in Fig. 1.

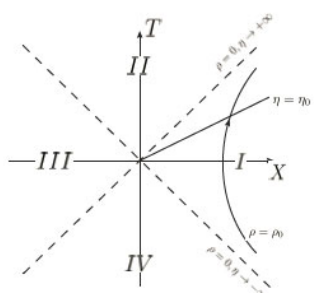

Figure 1: Casual structure of $\mathcal{M}_2$ in the Rindler form.

Note that

$$\begin{aligned} X = T(X>0): &\quad \eta\to\infty, \rho\to 0 \text{ with } \rho e^{\eta} \text{ finite} \\ X = -T(X>0): &\quad \eta\to-\infty, \rho\to 0 \text{ with } \rho e^{-\eta} \text{ finite} \\ X = T = 0: &\quad \rho\to 0, \text{any finite } \eta \end{aligned}$$

thus the horizon of a black hole $\rho = 0$ is mapped to the light cone $X = \pm T$. And near-horizon black hole geometry can be viewed as *Rindler* $\times S^2$ as shown in Fig. 2.

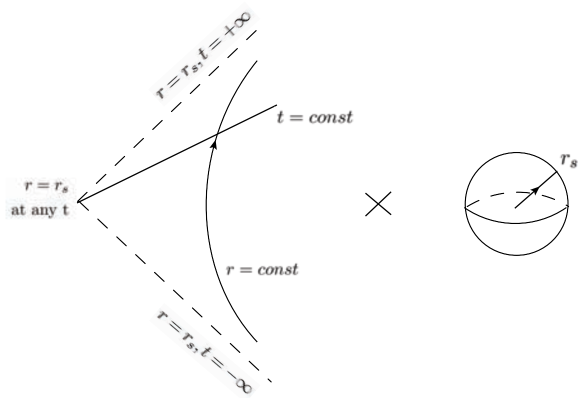

Figure 2: Near-horizon black hole geometry.

Remarks:

1. An observer at $r = const$ ($r \gtrsim r_s$) is mapped to and observer with $\rho = const$ in a Rindler patch, *i.e.* an observer in Minkowski spacetime following a hyperbolic trajectory

$$X^2 - T^2 = \rho^2 = const$$

   One thing to check here is such an observer has a constant proper acceleration

$$a = \frac{1}{\rho} = \frac{f'(r_s)}{2}\frac{1}{\sqrt{r-r_s}}$$

   And furthermore, the acceleration seen by O_∞ would be $a_\infty = a(r)f^{1/2}(r) = K$.

2. A free-fall observer near a black hole horizon is equal to an inertial observer in $\mathcal{M}_2$.

3. Rindler coordinates (ρ, η) become singular at $\rho = 0$, but using Minkowski coordinates (X, T), one could extent region I to the full Minkowski spacetime. Similarly, by changing to suitable coordinates (Kruskal coordinates), one can extend the Schwarzschild spacetime to four patches (Fig. 3).

   - Clearly, no information or observer in region II can reach region I (separated by a future horizon).
   - Region III and IV are related to I and II by time reversal. They do not exist for real black holes formed from gravitational collapse. Observer in region I cannot influence events in region IV (separated by a past horizon).
   - At $r = 0$, there is a black hole singularity, called curvature singularity which is space-like.

Penrose diagrams

In this section, we study Penrose diagrams, which are used to visualize the global causal structure of a spacetime.

In the following, we show the steps to draw a Penrose diagram of a spacetime. We start with the metric:

$$ds^2 = g_{ab}(x)dx^a dx^b$$

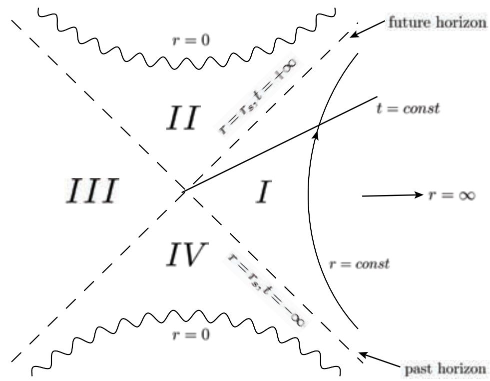

Figure 3: Schwarzschild blackhole geometry in Kruskal coordinates.

1. Find a coordinate transformation $x^a = x^a(y^\alpha)$ so that y^α has a finite the range (map the whole spacetime to a finite region).
2. Construct a new metric which is conformally related to the original one

$$d\tilde{s}^2 = \Omega^2(y) ds^2 = \tilde{g}_{\alpha\beta}(y) dy^\alpha dy^\beta$$

such that $\tilde{g}_{\alpha\beta}$ is simple. $d\tilde{s}^2$ and ds^2 have the same causal structure as null rays are preserved by conformal scalings.

Example:
(1+1)d Minkowski space

$$ds^2 = -dT^2 + dX^2 = -dUdV; \quad U = T - X, V = T + X$$

let $U = \tan u$, $V = \tan v$, then $u, v \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We define the following:

i_0: spatial infinity (X infinite, T finite)
i_+: time-like future infinity ($T \to \infty$, X finite)
i_-: time-like past infinity ($T \to -\infty$, X finite)
I_+: null future infinity (where all null rays end)
I_-: null past infinity (where all null rays start)

And we can label these points (lines) accordingly in the Penrose diagram for $\mathcal{M}_2$ (Fig. 4)

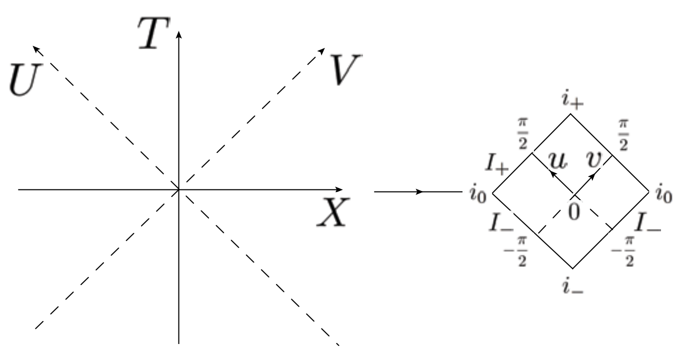

Figure 4: $\mathcal{M}_2$ Penrose diagram.

Another more interesting example is the Schwarzschild black hole

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega_2^2$$

1. We first consider (r, t) plane
2. Then we go to a coordinate system (Kruskal) which covers all four regions (analogue of U, V in Minkowski spacetime)
3. Next we make a coordinate transformation to make the new coordinate has a finite range

$$(U, V) \to (u, v)$$

In the end, we get the Penrose diagram of Schwarzschild black hole as shown in the right panel of Fig. 5

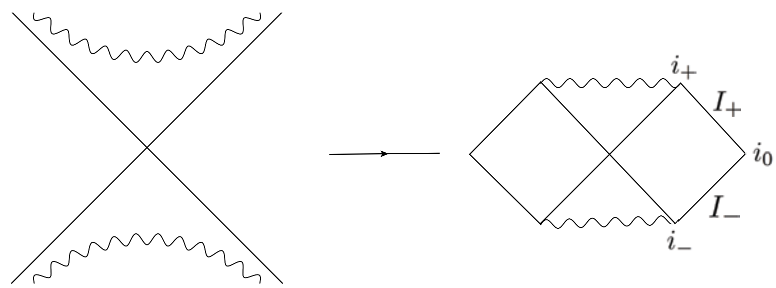

Figure 5: Schwarzschild black hole Penrose diagram.

1.2.3 black hole temperature

In this section, we will show that a black hole has a temperature, as viewed by a stationary observer outside the black hole. First we need to define the temperature. Recall that in QFT, to describe a system at finite temperature (T), we analytically continue to Euclidean signature, *i.e.* $t \to -i\tau$. And let τ to be periodic: $\tau \sim \tau + \hbar\beta$, with $\beta = \frac{1}{T}$. Notice that here we have also adapted the convention $k_B = 1$. Conversely if Euclidean continuation of a QFT is periodic in time direction, we can conclude that the QFT is at finite temperature. This is exactly what we are going to do to interpret the temperature of a black hole.

If we analytically continue the Schwarzschild metric to Euclidean signature with $t \to -i\tau$

$$ds_E^2 = f d\tau^2 + \frac{1}{f} dr^2 + r^2 d\Omega_2^2$$

Near the horizon

$$ds_E^2 = \rho^2 K^2 d\tau^2 + d\rho^2 + r^2 d\Omega_2^2 = \rho^2 d\theta^2 = d\rho^2 + r_s^2 d\Omega_2^2; \quad \theta = K\tau = \frac{\tau}{2r_s}$$

Note that the first two terms above describe a polar coordinates in Euclidean $\mathcal{R}^2$. This metric has a conical singularity unless θ is periodic in 2π, *i.e.* $\theta \sim \theta + 2\pi$. Since horizon is non-singular in Lorentzian signature, it should not be singular in Euclidean. Hence τ must be periodic

$$\tau \sim \tau + \frac{2\pi}{K}$$

Recall that t is the proper time for an observer at $r = \infty$, an observer at $r = \infty$ must feel a temperature:

$$T = \frac{1}{\beta} = \frac{\hbar K}{2\pi} = \frac{\hbar}{8\pi G_N m}$$

For an observer at some r, since $dt_{loc} = f^{1/2}(r) dt$, we have

$$T_{loc}(r) = \frac{\hbar K}{2\pi} f^{-1/2}(r)$$

This local temperature goes to ∞ as we approach the horizon, *i.e.* the black hole horizon is a very hot place for a stationary observer!

Similarly for Rindler spacetime

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 \to \eta = -i\theta ds_E^2 = \rho^2 d\theta^2 + d\rho^2$$

Since θ must be periodic in 2π, we define the local proper time: $d\tau_{loc}^2 = \rho^2 d\theta^2$. Then τ_{loc} must be periodic

$$T_{loc}^{Rindler}(\rho) = \frac{\hbar}{2\pi\rho} = \frac{\hbar a}{2\pi}$$

where we have used the relation $a = \frac{1}{\rho}$. So in Minkowski spacetime, an accelerated observer will feel a temperature proportional to its acceleration!

So now we have a quite simple derivation of black hole temperature as shown above, but one needs to dig further on the physics behind this simple picture.

8.821 / 8.871 String Theory and Holographic Duality
Fall 2014