8.512 Theory of Solids II
Spring 2009

Lecture 9: Superconductor Diamagnetism

In this lecture, we will apply linear response theory to the diamagnetism of a clean BCS superconductor.

9.1 Clean BCS Superconductor Diamagnetism at $T = 0$

9.1.1 General Considerations

Based on symmetry arguments, it is easy to see that when an isotropic system is placed in an external field, current always flows in the direction of the applied field. As a result, the paramagnetic current response tensor

$$R_{\mu\nu} = -i \langle 0 | \left[j^p_\mu(\vec{q}, t),\, j^p_\nu(-\vec{q}, 0) \right] | 0 \rangle \tag{9.1}$$

is *diagonal* for an isotropic system. Within the context of linear response theory, this definition of $R_{\mu\nu}$ yields

$$\langle j^p_\mu \rangle = -R_{\mu\nu} A_\nu(\vec{q}, \omega) \tag{9.2}$$

The total current also includes the diamagnetic piece

$$\langle j_\mu \rangle = \langle j^p_\mu \rangle + \langle j^d_\mu \rangle \tag{9.3}$$

$$\langle j^d_\mu \rangle = -\frac{ne^2}{mc^2} A_\mu \tag{9.4}$$

arising from the $\vec{A} \cdot \vec{A}$ term in the Hamiltonian.

Combining these terms, the total current to linear order in A_ν is given by

$$\langle j_\mu \rangle = -K_{\mu\nu}(\vec{q}, \omega) A_\nu(\vec{q}, \omega) \tag{9.5}$$

with total current response tensor

$$K_{\mu\nu} = R_{\mu\nu} + \frac{ne^2}{mc^2} \delta_{\mu\nu} \tag{9.6}$$

For normal metals, the constant diamagnetic current $-\frac{ne^2}{mc^2} A_\mu$ is exactly cancelled by part of the paramagnetic current. In a superconductor, however, this piece of the current survives. We begin by calculating the (diagonal) paramagnetic current response $R_{\mu\mu}$ of the BCS ground state.

$$R_{\mu\mu} = \sum_n |\langle n | j^p_\mu(\vec{q}) | 0 \rangle|^2 \left\{ \frac{1}{\omega - (E_n - E_0) + i\eta} - \frac{1}{\omega + (E_n - E_0) + i\eta} \right\} \tag{9.7}$$

9.1.2 Quick Review of BCS Theory

To calculate $R_{\mu\mu}$ for a clean (BCS) superconductor, we will need to evaluate the matrix element $\langle n | j^p_\mu(\vec{q}) | 0 \rangle$ on the eigenstates of the BCS effective Hamiltonian $H^{\text{BCS}}_{\text{eff}}$. In the basis of single-particle momentum-eigenstates, we have:

$$H^{\text{BCS}}_{\text{eff}} = \sum_{\vec{k},\sigma} \xi_{\vec{k}} c^\dagger_{\vec{k},\sigma} c_{\vec{k},\sigma} - \sum_{\vec{k}} \left(\Delta c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} + \Delta^* c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} \right) \tag{9.8}$$

where

$$\xi_{\vec{k}} = \epsilon_{\vec{k}} - \mu \tag{9.9}$$

and $c^\dagger_{\vec{k},\sigma}$ $(c^\dagger_{\vec{k},\sigma})$ is the creation (destruction) operator for an electron with momentum $\hbar\vec{k}$ and spin σ.

The first term of equation (9.8) is simply the free electron gas energy, relative to the chemical potential μ. The second term comes from mean field averaging-out the two destruction (creation) operators of the four operator interaction

$$\sum_{\vec{k},\vec{k}',\vec{q}} V_{\vec{k},\vec{k}'} c^\dagger_{\vec{k}'} c^\dagger_{\vec{k}} c_{\vec{k}} c_{\vec{k}'} \tag{9.10}$$

To diagonalize $H^{\text{BCS}}_{\text{eff}}$, we can change from the single particle momentum-eigenstate basis to the Bogoliubov quasiparticle basis states defined by the (unitary) transformation

$$\gamma_{\vec{k}\uparrow} = u_{\vec{k}} c_{\vec{k}\uparrow} - v_{\vec{k}} c^\dagger_{-\vec{k}\downarrow} \tag{9.11}$$

$$\gamma^\dagger_{-\vec{k}\downarrow} = v^*_{\vec{k}} c_{\vec{k}\uparrow} + u_{\vec{k}} c^\dagger_{-\vec{k}\downarrow} \tag{9.12}$$

where $u_{\vec{k}} \in \mathbb{R}$ and in general $v_{\vec{k}} \in \mathbb{C}$. To ensure unitarity, $|u_{\vec{k}}|^2 + |v_{\vec{k}}|^2 = 1$.

Because this transformation is simply a unitary transformation on single Fermion basis states, the resulting Bogliubov quasiparticle operators still satisfy the Fermionic anticommutation relations:

$$\{\gamma_{\vec{k},\sigma}, \gamma^\dagger_{\vec{k}',\sigma'}\} = \delta_{\vec{k}\vec{k}'} \delta_{\sigma\sigma'} \mathbb{1} \tag{9.13}$$

$$\{\gamma_{\vec{k},\sigma}, \gamma_{\vec{k}',\sigma'}\} = \{\gamma^\dagger_{\vec{k},\sigma}, \gamma^\dagger_{\vec{k}',\sigma'}\} = 0 \tag{9.14}$$

The inverse transformation is given by

$$c_{\vec{k}\uparrow} = u_{\vec{k}} \gamma_{\vec{k}\uparrow} + v_{\vec{k}} \gamma^\dagger_{-\vec{k}\downarrow} \tag{9.15}$$

$$c^\dagger_{-\vec{k}\downarrow} = -v^*_{\vec{k}} \gamma_{\vec{k}\uparrow} + u_{\vec{k}} \gamma^\dagger_{-\vec{k}\downarrow} \tag{9.16}$$

We have freedom in the choice of phase for $v_{\vec{k}}$, which is related to the winding number of the corresponding wave function. Assuming $v_{\vec{k}} \in \mathbb{R}$, substituting relations (9.15) and (9.16) into (9.8) we get

$$H^{\text{BCS}}_{\text{eff}} = \sum_{\vec{k},\sigma} E_{\vec{k}} \gamma^\dagger_{\vec{k},\sigma} \gamma_{\vec{k},\sigma} \tag{9.17}$$

with

$$E_{\vec{k}} = \sqrt{\xi_{\vec{k}}^{\,2} + |\Delta|^2} \tag{9.18}$$

$$u_{\vec{k}}^{\,2} = \frac{1}{2}\left(1 + \frac{\xi_{\vec{k}}}{E_{\vec{k}}}\right) \tag{9.19}$$

$$v_{\vec{k}}^{\,2} = \frac{1}{2}\left(1 - \frac{\xi_{\vec{k}}}{E_{\vec{k}}}\right) \tag{9.20}$$

Thus the BCS Hamiltonian is diagonalized by the Bogliubov states with spectrum $\{E_{\vec{k}}\}$.

9.1.3 Calculation of $R_{\mu\mu}$

To evaluate the matrix element in equation (9.7), we need the explicit form of the paramagnetic current operator. In second quantized notation in terms of the basis of single particle momentum-eigenstates,

$$j_\mu^p(\vec{q}) = -\frac{e}{m}\sum_{\vec{k},\sigma}\left(k_\mu + \frac{q_\mu}{2}\right) c^\dagger_{\vec{k}+\vec{q},\sigma} c_{\vec{k},\sigma} \tag{9.21}$$

Expanding out the sum over spins σ and letting $\vec{k} \to -(\vec{k}+\vec{q})$ for the $\sigma = \downarrow$ terms, we get

$$j_\mu^p(\vec{q}) = -\frac{e}{m}\sum_{\vec{k}}\left(k_\mu + \frac{q_\mu}{2}\right)\left(c^\dagger_{\vec{k}+\vec{q}\uparrow} c_{\vec{k}\uparrow} - c^\dagger_{-\vec{k}\downarrow} c_{-(\vec{k}+\vec{q})\downarrow}\right) \tag{9.22}$$

Now, we need to transform into the Bogoliubov basis by substituting relations (9.15) and (9.16) for $c_{\vec{k}\uparrow}$ and $c^\dagger_{-\vec{k}\downarrow}$. To do so, we will also need to make use of the Hermitian conjugates of (9.15) and (9.16):

$$c^\dagger_{\vec{k}\uparrow} = \; u_{\vec{k}}\gamma^\dagger_{\vec{k}\uparrow} + v^*_{\vec{k}}\gamma_{-\vec{k}\downarrow} \tag{9.23}$$

$$c_{-\vec{k}\downarrow} = -\,v_{\vec{k}}\gamma^\dagger_{\vec{k}\uparrow} + u_{\vec{k}}\gamma_{-\vec{k}\downarrow} \tag{9.24}$$

Inserting these relations, we get

$$\begin{aligned}
c^\dagger_{\vec{k}+\vec{q}\uparrow} c_{\vec{k}\uparrow} &= \left(u_{\vec{k}+\vec{q}}\gamma^\dagger_{\vec{k}+\vec{q}\uparrow} + v^*_{\vec{k}+\vec{q}}\gamma_{-(\vec{k}+\vec{q})\downarrow}\right)\left(u_{\vec{k}}\gamma_{\vec{k}\uparrow} + v_{\vec{k}}\gamma^\dagger_{-\vec{k}\downarrow}\right) \\
&= (u_{\vec{k}+\vec{q}}u_{\vec{k}})\gamma^\dagger_{\vec{k}+\vec{q}\uparrow}\gamma_{\vec{k}\uparrow} + (v_{\vec{k}}v^*_{\vec{k}+\vec{q}})\gamma_{-(\vec{k}+\vec{q})\downarrow}\gamma^\dagger_{-\vec{k}\downarrow} \\
&\quad + (u_{\vec{k}+\vec{q}}v_{\vec{k}})\gamma^\dagger_{\vec{k}+\vec{q}\uparrow}\gamma^\dagger_{-\vec{k}\downarrow} + (u_{\vec{k}}v^*_{\vec{k}+\vec{q}})\gamma_{-(\vec{k}+\vec{q})\downarrow}\gamma_{\vec{k}\uparrow} \\
&= (u_{\vec{k}+\vec{q}}u_{\vec{k}})\gamma^\dagger_{\vec{k}+\vec{q}\uparrow}\gamma_{\vec{k}\uparrow} - (v_{\vec{k}}v^*_{\vec{k}+\vec{q}})\gamma^\dagger_{-\vec{k}\downarrow}\gamma_{-(\vec{k}+\vec{q})\downarrow} + (v_{\vec{k}}v^*_{\vec{k}+\vec{q}})\{\gamma_{-(\vec{k}+\vec{q})\downarrow}, \gamma^\dagger_{-\vec{k}\downarrow}\} \\
&\quad + (u_{\vec{k}+\vec{q}}v_{\vec{k}})\gamma^\dagger_{\vec{k}+\vec{q}\uparrow}\gamma^\dagger_{-\vec{k}\downarrow} - (u_{\vec{k}}v^*_{\vec{k}+\vec{q}})\gamma_{\vec{k}\uparrow}\gamma_{-(\vec{k}+\vec{q})\downarrow}
\end{aligned} \tag{9.25}$$

and

$$\begin{aligned}
c^\dagger_{-\vec{k}\downarrow} c_{-(\vec{k}+\vec{q})\downarrow} &= \left(-v^*_{\vec{k}}\gamma_{\vec{k}\uparrow} + u_{\vec{k}}\gamma^\dagger_{-\vec{k}\downarrow}\right)\left(-v_{\vec{k}+\vec{q}}\gamma^\dagger_{\vec{k}+\vec{q}\uparrow} + u_{\vec{k}+\vec{q}}\gamma_{-(\vec{k}+\vec{q})\downarrow}\right) \\
&= (v^*_{\vec{k}}v_{\vec{k}+\vec{q}})\gamma_{\vec{k}\uparrow}\gamma^\dagger_{\vec{k}+\vec{q}\uparrow} - (v^*_{\vec{k}}u_{\vec{k}+\vec{q}})\gamma_{\vec{k}\uparrow}\gamma_{-(\vec{k}+\vec{q})\downarrow}
\end{aligned} \tag{9.26}$$

$$- (u_{\vec{k}} v_{\vec{k}+\vec{q}}) \gamma^\dagger_{-\vec{k}\downarrow} \gamma^\dagger_{\vec{k}+\vec{q}\uparrow} + (u_{\vec{k}} u_{\vec{k}+\vec{q}}) \gamma^\dagger_{-\vec{k}\downarrow} \gamma_{-(\vec{k}+\vec{q})\downarrow}$$

$$\begin{aligned} = &- (v^*_{\vec{k}} v_{\vec{k}+\vec{q}}) \gamma^\dagger_{\vec{k}+\vec{q}\uparrow} \gamma_{\vec{k}\uparrow} + (v^*_{\vec{k}} v_{\vec{k}+\vec{q}}) \{\gamma_{\vec{k}\uparrow}, \gamma^\dagger_{\vec{k}+\vec{q}\uparrow}\} \\ &- (v^*_{\vec{k}} u_{\vec{k}+\vec{q}}) \gamma_{\vec{k}\uparrow} \gamma_{-(\vec{k}+\vec{q})\downarrow} + (u_{\vec{k}} v_{\vec{k}+\vec{q}}) \gamma^\dagger_{\vec{k}+\vec{q}\uparrow} \gamma^\dagger_{-\vec{k}\downarrow} + (u_{\vec{k}} u_{\vec{k}+\vec{q}}) \gamma^\dagger_{-\vec{k}\downarrow} \gamma_{-(\vec{k}+\vec{q})\downarrow} \end{aligned}$$

Substituting these back into (9.22) and exercising our choice of $\{v_{\vec{k}}\} \in \mathbb{R}$, we get

$$j^p_\mu(\vec{q}) = -\frac{e}{m} \sum_{\vec{k}} \left(k_\mu + \frac{q_\mu}{2}\right) \Big\{ (u_{\vec{k}+\vec{q}} u_{\vec{k}} + v_{\vec{k}} v_{\vec{k}+\vec{q}}) \left(\gamma^\dagger_{\vec{k}+\vec{q}\uparrow} \gamma_{\vec{k}\uparrow} - \gamma^\dagger_{-\vec{k}\downarrow} \gamma_{-(\vec{k}+\vec{q})\downarrow}\right) \tag{9.27}$$

$$+ (u_{\vec{k}+\vec{q}} v_{\vec{k}} - u_{\vec{k}} v_{\vec{k}+\vec{q}}) \left(\gamma^\dagger_{\vec{k}+\vec{q}\uparrow} \gamma^\dagger_{-\vec{k}\downarrow} + \gamma_{\vec{k}\uparrow} \gamma_{-(\vec{k}+\vec{q})\downarrow}\right) \Big\}$$

Notice that the anticommutators $(v_{\vec{k}} v_{\vec{k}+\vec{q}})\{\gamma_{\vec{k}\uparrow}, \gamma^\dagger_{\vec{k}+\vec{q}\uparrow}\}$ and $(v_{\vec{k}} v_{\vec{k}+\vec{q}})\{\gamma_{-(\vec{k}+\vec{q})\downarrow}, \gamma^\dagger_{-\vec{k}\downarrow}\}$ are 0 for $\vec{q} \neq 0$ and cancel out for $\vec{q} = 0$ when $c^\dagger_{-\vec{k}\downarrow} c_{-(\vec{k}+\vec{q})\downarrow}$ is subtracted from $c^\dagger_{\vec{k}+\vec{q}\uparrow} c_{\vec{k}\uparrow}$. With the definitions

$$\ell_{\vec{k},\vec{k}+\vec{q}} \equiv u_{\vec{k}+\vec{q}} u_{\vec{k}} + v_{\vec{k}} v_{\vec{k}+\vec{q}} \tag{9.28}$$

$$p_{\vec{k},\vec{k}+\vec{q}} \equiv u_{\vec{k}+\vec{q}} v_{\vec{k}} - u_{\vec{k}} v_{\vec{k}+\vec{q}} \tag{9.29}$$

equation (9.27) becomes

$$j^p_\mu(\vec{q}) = -\frac{e}{m} \sum_{\vec{k}} \left(k_\mu + \frac{q_\mu}{2}\right) \Big\{ \ell_{\vec{k},\vec{k}+\vec{q}} \left(\gamma^\dagger_{\vec{k}+\vec{q}\uparrow} \gamma_{\vec{k}\uparrow} - \gamma^\dagger_{-\vec{k}\downarrow} \gamma_{-(\vec{k}+\vec{q})\downarrow}\right) \tag{9.30}$$

$$+ p_{\vec{k},\vec{k}+\vec{q}} \left(\gamma^\dagger_{\vec{k}+\vec{q}\uparrow} \gamma^\dagger_{-\vec{k}\downarrow} + \gamma_{\vec{k}\uparrow} \gamma_{-(\vec{k}+\vec{q})\downarrow}\right) \Big\}$$

The coefficients $\ell_{\vec{k},\vec{k}+\vec{q}}$ and $p_{\vec{k},\vec{k}+\vec{q}}$ are known as *coherence factors*. Note that each term in equation (9.30) conserves total spin projection along the z-axis, and has the effect of increasing the net total momentum by $\hbar\vec{q}$.

At $T = 0$, the BCS groundstate has no quasiparticle excitations. Thus

$$\gamma_{(\vec{k}+\vec{q})\uparrow} |\, 0\,\rangle_{\rm BCS} = \gamma_{-\vec{k}\downarrow} |\, 0\,\rangle_{\rm BCS} = 0 \tag{9.31}$$

The only term that survives is the term containing the double creation operator $\gamma^\dagger_{\vec{k}+\vec{q}\uparrow} \gamma^\dagger_{-\vec{k}\downarrow}$, yielding

$$R_{\mu\mu} = -\frac{e}{m} \sum_{\vec{k}} \left(k_\mu + \frac{q_\mu}{2}\right)^2 p^2_{\vec{k},\vec{k}+\vec{q}} \left\{ \frac{1}{\omega - (E_{\vec{k}} + E_{\vec{k}+\vec{q}}) + i\eta} - \frac{1}{\omega + (E_{\vec{k}} + E_{\vec{k}+\vec{q}}) + i\eta} \right\} \tag{9.32}$$

where the excitation energy $E_n - E_0 = (E_{\vec{k}} + E_{\vec{k}+\vec{q}})$ since each excited state comes from the double excitation $\gamma^\dagger_{\vec{k}+\vec{q}\uparrow} \gamma^\dagger_{-\vec{k}\downarrow}$.

In the DC limit, $\omega = 0$ and

$$R_{\mu\mu}(\omega = 0) \approx -\frac{e}{m} \sum_{\vec{k}} \left(k_\mu + \frac{q_\mu}{2}\right)^2 p^2_{\vec{k},\vec{k}+\vec{q}} \frac{1}{2\Delta} \tag{9.33}$$

since the largest contribution comes from the smallest energy excitations, which have energy 2Δ. In the $\vec{q} \to 0$ limit, however, $p_{\vec{k},\vec{k}+\vec{q}} \to 0$, which gives

$$R_{\mu\mu}(\omega = 0, \vec{q} \to 0) \longrightarrow \frac{0}{2\Delta} = 0 \tag{9.34}$$

Inserting this into the expression for the total current response tensor $K_{\mu\nu}$, we get

$$K_{\mu\nu}(\omega = 0, \vec{q} \to 0) = R_{\mu\nu}(\omega = 0, \vec{q} \to 0) + \frac{ne^2}{mc^2}\delta_{\mu\nu} = 0 + \frac{ne^2}{mc^2}\delta_{\mu\nu} \tag{9.35}$$

Thus in a superconductor, the diamagnetic current survives in contrast to the cancellation that occurs for a normal metal As a result, in this limit at $T = 0$

$$\langle \vec{j}\, \rangle = -\frac{ne^2}{mc^2}\vec{A} \tag{9.36}$$

This is a curious result, as the specific form of $\vec{A}$ is *gauge dependent*. In fact, the result is only true in the London Gauge in which $\nabla \cdot \vec{A} = 0$. How did this choice of gauge creep into our derivation?

It is possible that the gap Δ depends on $\vec{A}$. In this case, we would have to solve a new self-consistent BCS equation in the presence of the altered form. According to rotational invariance, any correction to Δ must take the form

$$\Delta = \Delta_0 + \vec{c} \cdot \vec{A} \tag{9.37}$$

where $\vec{c}$ is some vector relevant to the system. In the present case, the only relevant vector is $\vec{q}$. By choosing the London Gauge $\nabla \cdot \vec{A} = 0$ we ensure that $\vec{q} \cdot \vec{A} = 0$, thus guarantying the validity of the result just derived.

9.2 BCS Diamagnetism at Finite Temperatures

At finite temperatures $T > 0$, the quasiparticle state populations will in general be nonzero. Thus we expect all four terms of equation (9.30) to contribute to the matrix element in (9.7). Pulling all of these terms together, we get

$$\begin{aligned} R_{\mu\mu}(\vec{q}, \omega) = \frac{e^2}{m^2} \sum_{\vec{k}} \left(k_\mu + \frac{q_\mu}{2}\right)^2 \Bigg\{ &-2\,\ell^2_{\vec{k},\vec{k}+\vec{q}} \frac{f(E_{\vec{k}+\vec{q}}) - f(E_{\vec{k}})}{E_{\vec{k}} - E_{\vec{k}+\vec{q}} - \omega + i\eta} \\ &+ p^2_{\vec{k},\vec{k}+\vec{q}} \left[\frac{f(E_{\vec{k}+\vec{q}}) + f(E_{\vec{k}}) - 1}{\omega - (E_{\vec{k}} + E_{\vec{k}+\vec{q}}) + i\eta} - \frac{f(E_{\vec{k}+\vec{q}}) + f(E_{\vec{k}}) - 1}{\omega + (E_{\vec{k}} + E_{\vec{k}+\vec{q}}) + i\eta} \right] \Bigg\} \end{aligned} \tag{9.38}$$

If we now let $\omega = 0$ and take the limit $\vec{q} \to 0$, $p_{\vec{k},\vec{k}+\vec{q}} \to 0$ as before, but $\ell_{\vec{k},\vec{k}+\vec{q}} \to 1$ since $|u_{\vec{k}}|^2 + |v_{\vec{k}}|^2 = 1$. This yields

$$R_{\mu\mu}(\vec{q} \to 0, \omega = 0) = -2\, e^2 \sum_{\vec{k}} \left(\frac{k_\mu}{m}\right)^2 \frac{\partial f}{\partial E_{\vec{k}}} \neq 0 \tag{9.39}$$

Although the paramagnetic current response at finite temperature is nonzero, for low temperatures ($k_B T/\Delta \ll 1$) its contribution will be exponentially small and will not be sufficient to fully cancel the London diamagnetic current $\frac{ne^2}{mc^2}$. We can measure the degree to which the system retains its superconducting diamagnetic behavior by writing the total current response in terms of an effective "superfluid density" $\rho_s(T)$:

$$K_{\mu\mu} = -\frac{\rho_s(T)e^2}{mc^2} \tag{9.40}$$

In this "two-fluid" picture,

$$\rho_s(T) = \rho_s(0) - \rho_n(T) \tag{9.41}$$

where $\rho_n(T)$ is the "normal-fluid" density due to excited quasiparticles at temperature T. It is these excited quasiparticles that are responsible for the non-superconducting aspect of the system's behavior at finite temperatures. According to these definitions, we can identify

$$R_{\mu\mu}(\vec{q} \to 0, \omega = 0) = -2e^2 \sum_{\vec{k}} \left(\frac{k_\mu}{m}\right)^2 \frac{\partial f}{\partial E_{\vec{k}}} = \frac{e^2}{mc^2}\rho_n(T) \tag{9.42}$$

Near $T = 0$, the superfluid density behaves as $\rho_s(T) \propto e^{-\Delta/k_BT}$. Near the critical point, we get the mean-field result for the order parameter $\rho_s \approx |\Delta(T)|^2 \propto (T_c - T)$. As the temperature is raised from 0, the increase in quasiparticle excitations and decrease of the energy gap work together in a sort of "runaway process" to kill the superconductivity.

9.3 Superconductors with Vanishing Gaps

Although the BCS ground state only involves a thin skin of electrons near the Fermi surface, *all* electrons participate in the diamagnetism proportional to ne^2/mc^2. Thus an energy gap is *not* essential for superconductivity. In fact, there are many examples of gapless superconductors.

If fixed magnetic impurities are present, then electron-impurity scattering can break up BCS pairs through the spin-spin interaction. Due to impurity scattering, there is a finite density of states at the Fermi energy. The system can still be superconducting, however, with a significantly reduced superfluid density ρ_s. The net effect is that magnetic impurities reduce the critical temperature T_c.

Additionally, there are superconductors for which the gap Δ depends on direction (i.e. $\Delta = \Delta(\vec{k})$). Along some directions $\{\vec{k}'\}$ the gap may vanish. Such directions are called nodes. For d-wave superconductors, $\Delta(\vec{k}) \propto \cos 2\theta$, which has four nodes at $\theta_n = (2n+1)\pi/4$. Nonetheless, superconducting behavior is still observed.

9.4 Coherence Factors

In addition to the coherence factors $\ell_{\vec{k},\vec{k}+\vec{q}}$ and $p_{\vec{k},\vec{k}+\vec{q}}$ that arose in our consideration of the BCS matrix elements of the paramagnetic current operator, there are several other analogous coherence factors that arise from the consideration of the matrix elements of other operators. These coherence factors are ubiquitous in problems involving superconductors.

9.4.1 Ultrasonic Attenuation

One such example is the calculation of ultrasonic attenuation in a superconductor — what is the lifetime of a phonon sent through a superconductor? In a normal metal, the phonon can scatter an electron into a higher energy state, affecting a so-called particle-hole excitation. At $T = 0$, however, the lack of any quasiparticles and $\hbar\omega < \Delta$ imply that there is no way for the system to absorb energy from the photon. As T increases and the normal-fluid density increases as well, absorption is possible and phonon attenuation increases monotonically to the normal metal value at $T = T_c$.

In this situation, the relevant operator is the electron-phonon coupling

$$c^{\dagger}_{\vec{k}+\vec{q},\sigma} c_{\vec{k},\sigma} \left(b_{\vec{q}} + b^{\dagger}_{-\vec{q}} \right) \tag{9.43}$$

By substituting the relations for $c^{\dagger}_{\vec{k}+\vec{q},\sigma}$ and $c_{\vec{k},\sigma}$ in terms of the Bogoliubov quasiparticle creation/destruction operators, a different set of coherence factors is obtained.

9.4.2 Nuclear Spin Relaxation Rate $(1/T_1)$

Another interesting effect to examine is the longitudinal relaxation rate for a magnetic nucleus embedded in a metallic sample. The interaction has the form

$$H_{IS} = A\hat{\vec{I}} \cdot \hat{\vec{S}} \tag{9.44}$$

where $\hat{\vec{I}}$ and $\hat{\vec{S}}$ are the nuclear and conduction electron spin operators, respectively.

One effect of electron-impurity scattering is that a conduction electron can undergo a spin-flip and be sent outside the Fermi sphere. If there is an electron spin-flip, the magnetic nucleus must also undergo a flip of its own. Thus relaxation of the magnetic nucleus is expected.

If the calculation is carried out, one discovers the *Korringa Law*

$$\frac{1}{k_B T T_1} = \text{const} \tag{9.45}$$

for the normal metal state.

Below T_c, the relaxation rate rises to a *maximum* before decaying away to 0 as $T \to 0$. This peak is called a Hebel-Slichter peak, and results from the small enhancement of the density of states near the gap edge that makes up for the loss of density in the gap region. It is a bit surprising that a similar peak is *not* observed for the phonon attenuation. However, the coherence factors that arise in the ultrasonic attenuation calculation exactly cancel this effect, giving rise to the observed monotonic behavior. For more on this topic, see the books by Schrieffer or Phillips.