8.512 Theory of Solids II
Spring 2009

**8.512 Theory of Solids II** **Problem Set 7** **Due April 6, 2009**

1. Optical conductivity of disordered superconductors.

   Following our discussion of disordered metals, the optical conductivity of a disordered superconductor is given by the Kubo formula (which is easily derived by considering the rate of absorption of electromagnetic radiation):

$$\sigma(q \to 0, \omega) = \frac{\pi}{\omega} \frac{1}{\Omega} \sum_n |\langle 0| \int d\boldsymbol{r} j_x^P(\boldsymbol{r}) |n\rangle|^2 \sigma(\omega - (E_n - E_0)) \tag{1}$$

   where $\Omega$ is the volume. The paramagnetic current operator is written in the exact eigenstate representation as:

$$\int d\boldsymbol{r} j_x^P(\boldsymbol{r}) = e \sum_{\alpha,\beta,\sigma} v_{\alpha,\beta} c^\dagger_{\beta,\sigma} c_{\alpha,\sigma} \quad , \tag{2}$$

   and

$$v_{\alpha,\beta} = \frac{1}{m} \int d\boldsymbol{r} \phi^\alpha_\beta \frac{\nabla_x}{i} \phi_\alpha \tag{3}$$

   is the velocity matrix elements between exact eigenstates of the Hamiltonian $\mathcal{H}_1$, which describes free fermions with a disordered potential

$$\mathcal{H}_1 \phi_\alpha = \varepsilon_\alpha \phi_\beta \quad . \tag{4}$$

   In Eq. (1), $|0\rangle$ and $|n\rangle$ are the ground and excited states of the BCS mean field Hamiltonian in the presence of disorder.

   (a) Using the Bogolinbov transformation, show that

$$\sigma(q \to 0, \omega) = \frac{e^2}{\omega} \frac{\pi}{\Omega} \sum_{\alpha\beta} \overline{(u_\alpha v_\beta - v_\alpha u_\beta)^2 |v_{\alpha\beta}|^2 \sigma(\omega - E_\alpha - E_\beta)} \tag{5}$$

   where

$$\begin{aligned} u_\alpha^2 &= \frac{1}{2}\left(1+\frac{\xi_\alpha}{E_\alpha}\right) && (6)\\ v_\alpha^2 &= \frac{1}{2}\left(1-\frac{\xi_\alpha}{E_\alpha}\right) && (7)\\ E_\alpha &= \sqrt{\xi_\alpha^2+\Delta^2}\ ,\xi_\alpha=\varepsilon_\alpha-\mu \quad . && (8)\end{aligned}$$

By defining

$$f(\xi,\xi') = \frac{1}{\Omega}\sum_{\alpha\beta}\overline{|v_{\alpha\beta}|^2\sigma(\xi-\xi_\alpha)\sigma(\xi'-\xi_\beta)} \quad , \tag{9}$$

show that Eq. (5) can be written as

$$\sigma(q\to 0,\omega) = \frac{e^2}{\omega}\prod\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\xi'(uv'-vu')^2 f(\xi,\xi')\delta(\omega-E-E') \tag{10}$$

where the relation between $u$, $v$, $E$ and $\xi$ is given by Eqs. (6–8).

(b) Show that

$$(uv'-vu')^2 = \frac{1}{2}\left(1-\frac{\xi\xi'}{EE'}-\frac{\Delta^2}{EE'}\right) \quad . \tag{11}$$

(c) Note that $f(\xi,\xi')$ depends only on the normal state properties. Indeed it appeared in our treatment of disordered metals. By factorizing the impurity average, argue that $f(\xi,\xi')$ can be approximated by a constant for small $|\xi|$ and $|\xi'|$ , i.e. for energies near the Fermi level. (More accurately, $|\xi-\xi'|\ll\frac{1}{\tau}$. Amuse yourself by trying to point out at what step in the argument was this condition imposed.) By taking the limit $\Delta\to 0$, show how the expression we derived for the normal state conductivity $\sigma_N$ can be recovered. (Note how the spin sum is magically included.)

(d) Show that Eq. (10) simplifies to

$$\sigma(q\to 0,\omega) = \sigma_N\frac{1}{\omega}\int_0^{\infty}d\xi\int_0^{\infty}d\xi'\left(1-\frac{\Delta^2}{EE'}\right)\sigma(\omega-E-E') \quad . \tag{12}$$

This is known as the Mattis-Bardeen formula. By changing the integration vari able from $\xi$ to $E$, Eq. (12) reduces to a one-dimensional integral which can be done numerically or by mathematics. Sketch $\sigma/\sigma_N$ and comment on the key features. We emphasize that the Mattis-Bardeen formula is valid only for disordered superconductors and for $\Delta\tau \ll 1$.