8.512 Theory of Solids II
Spring 2009

8.512 Theory of Solids II Problem Set 4 Due March 9, 2009

1. This problem reviews the Boltzmann equation and compares the result with the Kubo formula. For a derivation of the Boltzmann equation, read p.319 of Ashcroft and Mermin.

   (a) Consider an electron gas subject to an electron field

$$\vec{E}\,(\vec{r},t) = \vec{E}_0\, e^{i(\vec{q}\cdot\vec{r}-i\omega t)} \tag{1}$$

   The Boltzmann equation in the relaxation time approximation is

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{r}}\cdot \vec{v} + \frac{\partial f}{\partial \vec{k}}(-e)\,\vec{E} = -\frac{f-f_0}{\tau} \tag{2}$$

   where f_0 is the equilibrium distribution

$$f_0(\epsilon) = \frac{1}{e^{\beta\epsilon}+1} \tag{3}$$

   Write

$$f(\vec{r},\vec{k},t) = f_0(\epsilon_k) + \Phi(\vec{k})e^{i\vec{q}\cdot\vec{r}-i\omega t} \tag{4}$$

   and work to first order in $\Phi(\vec{k})$ and $\vec{E}$. Show that the conductivity is given by

$$\sigma(\vec{q},\omega) = \frac{e^2}{4\pi^3}\int d\,\vec{k}\;\frac{\tau(\hat{e}\cdot\vec{v})^2}{1-i\tau(\omega-\vec{q}\cdot\vec{v})}\left(-\frac{\partial f_0}{\partial \epsilon_k}\right) \tag{5}$$

   where $\hat{e}$ is the unit vector in the direction of $\vec{E}_0$.

   (b) A simple way to derive the Kubo formula is to compare the energy dissipation rate σE_0^2 with the rate of photon absorption. At finite temperature, we need to include both absorption and emission processes. Show that for free electrons (including spin)

$$\sigma'(q,\omega) = \frac{2e^2}{m^2}\frac{1}{V}\sum_{\alpha,\beta}|<\beta|e^{i\vec{q}\cdot\vec{r}}\hat{e}\cdot\vec{p}\,|\alpha>|^2\frac{(f_0(E_\alpha)-f_0(E_\beta))}{(E_\beta-E_\alpha)/\hbar}\delta\left(\hbar\omega-(E_\beta-E_\alpha)\right) \tag{6}$$

   Using the Kramers-Kronig relation, show that the complex conductivity is

$$\sigma(\vec{q},\omega) = \frac{2e^2}{m^2}\frac{1}{V}\sum_{\alpha,\beta}\frac{|<\beta|e^{i\vec{q}\cdot\vec{r}}\hat{e}\cdot\vec{p}\,|\alpha>|^2}{(E_\beta-E_\alpha)\hbar}\frac{(-i)\left(f_0(E_\alpha)-f_0(E_\beta)\right)}{(E_\beta-E_\alpha-\hbar\omega-i\eta)} \tag{7}$$

(c) For $|q| \ll k_F$, show that Eq.(7) reduces to Eq.(5) under the assumptions that $|\alpha>, |\beta>$ are plane waves and η is identified with $\frac{1}{\tau}$.

2. Equation (5) in Problem 1 is valid for any relation between $\vec{q}$ and $\hat{e}$. In an isotropic material the response can be separated into the longitudinal ($\vec{q} \parallel \hat{e}$) and transverse parts ($\vec{q} \perp \hat{e}$). The latter is appropriate for the propagation of electromagnetic waves.

(a) For $T \ll \epsilon_F$, show that the transverse conductivity can be written as an integration over the Fermi surface.

$$\sigma_\perp(\vec{q}, \omega) = \frac{\sigma_0}{1 - i\omega\tau} \frac{3}{4} \int_{-1}^{1} dx \frac{1 - x^2}{1 + sx} \tag{8}$$

where

$$s = \frac{iqv_F\tau}{1 - i\omega\tau} \tag{9}$$

In Eq.(8) $\sigma_0 = ne^2\tau/m$ is the DC Boltzmann conductivity and the integration variable x stands for $\cos\theta$ in an integration over the Fermi surface.

(b) The integral in Eq.(8) can be done analytically. For our purposes, find the small $|s|$ and large $|s|$ limits. The small $|s|$ limit is the Drude conductivity while the large $|s|$ limit is called the "extreme anomalous region." It describes the situation when the electron mean free path ℓ is much greater than the wavelength of light. Note that it is reduced from σ_0 by the factor $1/(q\ell)$. Produce a simple argument to show that this reduction factor can be understood on the basis of kinetic theory of classical particles. (**Hint:** Consider a low frequency transverse electromagnetic wave. For $q\ell \ll 1$, all the electrons can absorb energy from the electric field. However, for $q\ell \gg 1$, only a fraction travelling almost parallel to $\hat{e}$ can do so. The argument was first given by Pippard.)