8.512 Theory of Solids II
Spring 2009

8.512 Theory of Solids II **Problem Set 2** **Due February** 23, 2009

1. Estimate the mean free path for plasmon production by a fast electron through a metal by the following steps:

   (a) For small q it is a good approximation to assume that $\varepsilon^{-1}(q,\omega)$ is dominated by the plasmon pole, i.e.

   $$Im - \frac{1}{\varepsilon(q,\omega)} \approx A(q)\delta(\omega - \omega_{pl}) \ .$$

   Determine the constant $A(q)$, using the f-sum rule.

   (b) Using (a), write down an expression for the probability of scattering into a solid angle Ω by emitting a plasmon.

   (c) Estimate the mean free path for plasmon emission. Put in some typical numbers (electron energy = 100 KeV, etc.)

2. Dielectric constant of a semiconductor.

   (a) In a periodic solid, show that the dielectric response function is given within the random phase approximation by

   $$\varepsilon(q,\omega) = 1 + \frac{4\pi e^2}{q^2}\sum_k \frac{|\langle \mathbf{k}+\mathbf{q},\beta|e^{i\mathbf{q}\cdot\mathbf{r}}|\mathbf{k},\alpha\rangle|^2 (f(\varepsilon_k^\alpha) - f(\varepsilon_{\mathbf{k}+\mathbf{q}}^\beta))}{\varepsilon_{\mathbf{k}+\mathbf{q}}^\beta - \varepsilon_{\mathbf{k}}^\alpha - \omega - i\eta} \tag{1}$$

   where the sum over $\mathbf{k}$ is over the first Brillouin zone and ε_k^α is the energy of band α.

   (b) We will evaluate Eq. (1) in an approximate way for a semiconductor in the limit $\omega = 0$ and $q \to 0$. Argue that the energy denominator in Eq. (1) may be replaced by an energy scale Δ given by the average energy gap. To estimate the numerator, derive the following theorem:

   $$\sum_b (\varepsilon_b - \varepsilon_a)|\langle b|e^{i\mathbf{q}\cdot\mathbf{r}}|\rangle a|^2 = \frac{\hbar^2 q^2}{2m} \tag{2}$$

   where $|a\rangle$ and $|b\rangle$ are the eigenstate of a Hamiltonian $\mathcal{H}$ with a kinetic energy term $-\hbar^2\nabla^2/2m$. This is a generalization of the f-sum rule in atomic physics. It is proven by evaluating the expectation value of

   $$\left[[H, e^{i\mathbf{q}\cdot\mathbf{r}}], e^{-i\mathbf{q}\cdot\mathbf{r}}\right]$$

   in the state $|a\rangle$. [We assume H obeys time reversal symmetry, i.e. $\psi_\alpha(r)$ and $\psi_\alpha^*(r)$ are both eigenfunctions with energy E_α.]

(c) By making the further approximation that the energy difference in Eq. (2) may be replaced by the energy scale Δ, show that for a semiconductor

$$\varepsilon(q \to 0, \omega = 0) = 1 + \left(\frac{\hbar\omega_{pl}}{\Delta}\right)^2$$

where ω_{pl} is the plasma frequency. Estimate ε for Si and Ge and compare with experiment.