8.511 Theory of Solids I **Problem Set 9** **Due November 23, 2004**

Consider the two-site Hubbard model

$$H = t \sum_{\sigma} \left(c^{\dagger}_{1\sigma} c_{2\sigma} + h.c. \right) + U \sum_{i=1,2} n_{i\uparrow} n_{i\downarrow}$$

where $c^{\dagger}_{i\sigma}$ creates an electron with spin σ on site i and $n_{i\sigma} = c^{\dagger}_{i\sigma} c_{i\sigma}$ is the number operator of electrons with spin σ on site i.

1. Write down the basis set (using creation operators) which spans the Hilbert space with 1, 2, 3, and 4 electrons. In this basis write down the Hamiltonian matrix. What are the eigenvalues and eigenvectors in the case of 1, 3, and 4 electrons?

2. Consider the case of two electrons. Calculate the eigenvalues and degeneracies exactly. In the Case $U \gg t$, show that the lowest two eigenvalues and degeneracies match those of the spin $\frac{1}{2}$ Heisenberg model $J\mathbf{S}_1 \cdot \mathbf{S}_2$, where $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}$ are the Pauli matrices. What is the value of J?