8.511 Theory of Solids I **Problem Set 11** **Due December 9, 2004**

1. (a) Start with the reduced Hamiltonian

$$\mathcal{H} = H_0 - V_0 \sum_{\mathbf{k},\mathbf{k}'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}$$

where $H_0 = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma}$. Show that $W \equiv < \Psi|H - \mu N|\Psi >$ is equal to

$$W = \sum_{\mathbf{k}} 2(\varepsilon_{\mathbf{k}} - \mu)|v_{\mathbf{k}}|^2 - V_0 \sum_{\mathbf{k},\mathbf{k}'} u_{\mathbf{k}'} v^*_{\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}}, \tag{1}$$

where $|\Psi>$ is the BCS wavefunction.

(b) Derive the BCS self-consistent equation

$$1 = N(0)V_0 \int_0^{\omega_D} d\xi \frac{1}{\sqrt{\xi^2 + \Delta^2}} \tag{2}$$

by directly minimizing the function W with respect to $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ subject to the constraint $u^2_{\mathbf{k}} + |v_{\mathbf{k}}|^2 = 1$. It is useful to parametrize $u_{\mathbf{k}} = \cos\theta_{\mathbf{k}}$ and $v_{\mathbf{k}} = \sin\theta_{\mathbf{k}}$, and show that $\theta_{\mathbf{k}}$ obeys the form

$$\tan 2\theta_{\mathbf{k}} = \frac{\Delta}{\xi_{\mathbf{k}}}$$

where

$$\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$$

.

(c) At finite temperature, Eq. (2) becomes

$$1 = N(0)V_0 \int_0^{\omega_D} d\xi \tanh\left(\frac{\sqrt{\xi^2 + \Delta^2}}{2kT}\right) \frac{1}{\sqrt{\xi^2 + \Delta^2}}$$

Show that

$$\Delta(T) = \Delta_0 \left[1 - a e^{-\Delta_0/T}\right]$$

for $T \ll \Delta_0$ where a is some constant times $\sqrt{T}$.