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PROFESSOR: We're going to continue our discussion of correlation functions, and in particular, g_2 of 0. If you think I'm focusing a lot on one quantity, g_2 of 0, that's correct. But I have to tell you, I have been involved in so many discussions with people who looked at g_2 of 0 with both Einstein condensates when Bose-Einstein condensates in [INAUDIBLE] were discovered here at MIT by [INAUDIBLE] and collaborators. There was some mystery about, in essence, the g_2 of 0 function.

So a lot of physics actually goes into it. And I've learned a lot from my own research and through discussions. And in this unit, I want to summarize it. So the confusion sometimes comes because people use classical pictures, quantum pictures, and they may not see the relation. So let's continue the discussion.

I want to go through four different views of the same physics that you will recognize that they're interrelated. The first one, just to repeat that, is if you have light which is Gaussian intensity fluctuations, light from random sources. I think if you would look at the twinkle of starlight, if you look at the light from a light bulb, because of the central limit theorem, you will find that the intensity that the electric field has a Gaussian distribution, and the intensity in exponential distribution.

And if you then ask, what is g_2 of 0, you'll find it's 2. Because for an exponential distribution, the average of the intensity squared is two times the intensity, the average of the-- what did I say. This average of the squared intensity is two times the average of the intensity squared.

Before I forget it, one student asked me after class, I'm discussing g_2 of 0, but what about g_2 of tau? Well, my understanding is that almost all of the physics is in g_2 of 0. And there is a simple number, 1 or 2, in those special cases.

What happens is, if you have completely independent sources, if you completely independent statistics, then the probability of finding a second particle, a second photon, is independent of the first. So it means g_2 of τ would be unity. This is the case for an uncoordinated system.

But if you have g_2 of 0 which is higher, this means correlation. This means some form of coherence. And usually, the correlated value simply decays to the uncorrelated value with a characteristic coherence time. And this characteristic coherence time can be inhomogeneous, technical. It can be limited by the Fourier transform of a light pulse, the monochromaticity, the bend which $\Delta\omega$ is $1/\tau$ over τ .

So it has all of the usual suspects for what limits the coherence time. So therefore in general, we're not learning something fundamental in looking at g_2 of τ . We're just learning the usual spiel about correlation times and coherence times.

The second angle I gave you with the brief interference is something which is deeply insightful. It shows you the following. If you have a plane wave, the intensity is constant. If you have two plane waves which interfere, you create density fluctuations, because the interference pattern can be constructive and can be destructive.

So therefore, if the intensity fluctuates, then i square average is no longer i average square. And what happens is because of the nature of interference, because the average of cosine square kx is-- well, there are factors of $1/2$ and 2 which just come out of cosine and sine square function. You get exactly g_2 of 2 .

But that should already tell you something, which will be important for both quantum particles, and light, and the classic limit. The moment if you have light or particles in one single mode, you can't have any form of interference. Only two modes can interfere. And you will always get a g_2 function of 1 . But we'll come to that in a second.

Any questions at that point? Yes.

AUDIENCE: I thought that when we talked about the g_2 function several weeks ago, you said that if we had a single mode thermal light, it would have the g_2 function of 2. And that's therefore, all we were using for lasers when we used them in experiments is the fact that it's single mode, not the coherence in the g_2 function.

PROFESSOR: You have a good-- that's a good point, Cory. Let's just try to connect the two pictures. Slowly, slowly. What did we discuss?

This question is great for me to repeat something. When we looked at black body radiation, I wrote down the partition function. And I said I'm only considering a single mode.

And then, we have of course in thermal equilibrium a Boltzmann type, or Bose-Einstein type, distribution of finding certain numbers, number of photons populating the single mode. And that resulted in a g_2 function of 2.

AUDIENCE: It's not really single mode though, right? Because you're summing over many frequencies.

PROFESSOR: No, it was single frequency. I have to think about it. That's a really good question.

Right now, I would quickly say, in thermal equilibrium-- I'll give you partial answer, and when I think about it, I'll give you full answer on Friday at the beginning of the next class-- I think would be calculated where the thermal fluctuations in the intensity. And the intensity fluctuations, and those intensity fluctuations. Those intensity fluctuations result in a g_2 function of 2. That's correct.

Maybe what I want you to say here is, if you have two modes which have all maybe single occupation or constant occupation, and they interfere, then they create a spatial interference pattern if the phase between the two modes is random, which gives rise to additional fluctuation. So most likely, the two fluctuations are independent. One is the fluctuations because of the interference of two modes.

But when I say the single mode has a g_2 function of 1, I meant actually a single mode with constant amplitude. I'll double check and give you a more complete

answer on Friday. Other questions? OK.

The third view is classical versus quantum statistics. And I think this really shows that we need at least some quantumness to find correlations. Let's assume we have n particles. And we have N possible states.

And if it's a classical system, let's assume we want to find one particle in a certain state. We have a big box and want to-- we have a big box, have a small sub-volume, and say what is the probability of finding one particle here?

So this probability to find one particle is P_1 is small n over big N . The probability to find two particles. If you have non-interacting classical particles, well, it's simply P_1 squared. Independent classical particles, they don't care what they are doing. You grab into a sub-volume, which is maybe phase space cell.

And therefore, your quantum state, you have probability P of finding one. But since each particle moves around independently, the probability to find two particles is just P squared. It's a little bit like if you toss two coins, one with your left and one with your right, and you ask what is the probability to find a head or tail, and you simply multiply the probabilities.

But now, following reasonings which Bose and Einstein introduced, you want to use counting statistics, which takes into account the indistinguishability of particles. So if we go from the classical distribution to the distribution of indistinguishable particles, then the classical probability P_1 is reduced by n factorial, because we are not-- let me put it this way.

The number-- I don't want to go through a mathematical argument. If you grab one particle, it can be the probability to find one particle is represented by a microconfiguration classically. And each configuration when you permute particles counts as independent. But if you have quantum indistinguishability, you're not counting permutations as an independent configuration. And therefore, you have a reduction by n factorial.

If you look at the probability of finding two particles, the reduction is-- this is just the

counting statistics n factorial by 2. So therefore, you will find that classically, that quantum mechanically for bosons, is two times. The probability to find two bosons in one quantum state is two times the probability squared to find one boson. So this is just counting statistics.

This can be applied to conditions in a Bose-Einstein in a Bose gas. When you have inelastic collisions, spin relaxation, dipolar collisions, two body collisions, two-body collisions have a rate γ_2 , which is proportional to the probability of finding two particles at the same time at the same location. And that brings in the g_2 function.

Whereas three-body collisions, γ_3 , reflects the third order correlation function g_3 , which is defined in an analogous way. So if you now compare at the same density the two-body rate coefficient, like the same density means we are looking at the rate of two-body collisions.

And we compare between two-body collision in a thermal cloud, which is a g_2 factor of two, a thermal cloud of bosons. Whereas a Bose-Einstein condensate is a constant occupation of particles in one mode and has therefore a g_2 factor of 1, exactly as [INAUDIBLE] of the Bose-Einstein condensate is for particles for meta waves, but the laser is for light. Whereas, if you look at three-body collisions, those who do experiments with Bose-Einstein condensates will know that usually the lifetime of Bose-Einstein condensates is limited by three-body collisions.

Well, I didn't derive it, but actually yes, we have the Gaussian statistics for randomness. For Gaussian statistics, the n -body collision function has a factor of n factorial. So therefore, three-body collisions scale with 3 factorial, and two-body collisions with 2 factorial.

So in other words, you have a thermal cloud, you have a Bose-Einstein condensate at the same density. But what matters for two-body collision is not the average density. It is the average of n square.

And the fluctuations in the thermal cloud because of the g_2 function are two times enhanced compared to Bose-Einstein condensate. And therefore, you find that at

the same density, you have more loss. You have a higher rate of two and three-body collisions.

Some of this was not-- became better understood soon after Bose-Einstein condensates were realized. The Boulder group studied three-body collisions. And for two-body collisions, myself in the post-doc clarified the situation that even the mean field energy, which everybody had measured before, requires two particles to interact with short range interaction is therefore proportional to g^2 . And therefore, when people had determined the mean field energy of a condensate without really knowing they had already determined the g^2 function of Bose-Einstein condensates.

OK, so this is pretty much the counting statistic which you do in undergraduate class when you derive the three different statistics-- Boltzmann statistics, Bose-Einstein statistics, Fermi statistics. And it gives you the result of that is also for bosons, you have the factor of 2.

For fermions, of course, you get 0. The average between bosons 2 and fermions 0 is 1. And this is the classical statistics. Any questions?

Then let me finish this discussion with a quantum mechanical description. If you describe the detection or collision of two particles in two modes. So you have mode.

A mode operator, annihilation operator a_1 you and a_2 . But if you are asking what happens if I measure two particles, you annihilate two particles with an operator a_1 and a_2 . But then, because of the indistinguishability of particles, you have to consider an exchange term, which is $a_2 a_1$. So whenever you detect two particles in two different modes, your signal is proportional to something related to $a_1 a_2$ plus $a_2 a_1$.

This extra exchange term gives you an extra factor of 2 for bosons, which is exactly what appears in the g^2 function. And, of course, if you have an exchange term for fermions, you get 0. And this, of course, leads to the antibunching, or the g^2 value of 0 for fermions.

However, and this is obvious, if you have a single mode, then the only operator to

detect two particles is a_1 times a_1 . And this has no exchange term. And this is the situation of the laser and the Bose-Einstein condensate. And that means they have a g_2 function of 1.

So Cory, this is another argument why single mode occupation has a g_2 function of 1. But I have to reconcile it with your question. Right now, I think there may be difference between a canonical ensemble and grand canonical ensemble.

In a canonical ensemble, we have [INAUDIBLE] the total particle number is fixed. But in a grand canonical it fluctuates. And maybe this leads to additional fluctuations for the case of a single mode chaotic light. But I will have to think about it, your question.

But here is the same argument why if you occupy only a single mode-- if you look, for instance, at a Bose-Einstein condensate, you do not have the second exchange term. And the missing exchange term just propagates through the equations and gives you the g_2 factor, the g_2 function of 1.

OK so let me sort of wrap it up. I've given you different angles at the g_2 function. Some are simply classical intensity distribution. Some is simply interference. Another aspect is Poissonian counting statistics.

My understanding and my interpretation is they look as different as they can possibly look, but they reflect the same physics. Because interference of light only happens because light consists of photons. And photons are bosons.

So the classical interference and the quantum mechanical counting of bosons lead not for random reasons, but lead for profound reasons to the same result. And chaotic light, which seems to be determined by just random fluctuation, well, the random fluctuations if you have single mode light, a thermal distribution of single mode photons, comes from random phases. And these random phases lead to random interference. And so we are back to interference.

Finally, let me give you my view on the measurements of g_2 . When you look through the literature, you find the famous Henry [INAUDIBLE] experiment. In one of

your homework assignments, you looked at a seminal experiment where people dropped atoms out of cold atom clouds and measured the g^2 function for cold atoms. Sometimes it's confusing when you directly compare the two experiments. But let me try to give you a common description, or the common denominator, between all these experiments.

You can say that all experiments to measure the two particle correlation function is about comparing the probability of finding two particles with a probability of finding one particle. And well, you restrict your measurement, your detection of particles, to either one quantum state, one mode of the electromagnetic field, one coherence length, or coherence volume, of light. Or if you use the semi-classical argument for the description of particles, you take one phase space cell.

This is one phase space cell, one mode, one coherence volume, one quantum state, is sort of the different definitions of what a quantum state is, or what a wave packet is with Heisenberg limited uncertainty. You cannot define a particle in phase space to within better than the phase space volume of a quantum state, which is h square. Or is it h bar? I don't think there is a bar-- h square.

And so this is how you can relate wave packets to quantum states to phase space volume. It's pretty much mathematically I could say in different basis sets, in different basis sets, it is one quantum state.

But you can use wave packets. You can use time dependent description. But what I said here-- one quantum state, one mode, one coherence volume-- this is as well as you can define a particle or photon to be by fundamental reasons by Heisenberg's uncertainty relation.

AUDIENCE: What do you mean by one coherence volume?

PROFESSOR: What?

AUDIENCE: What do we mean by one coherence volume?

PROFESSOR: Well, if you have a laser beam which propagates, and it is TEM00, you would say

the transverse coherence volume is the size of the laser beam but then the longitudinal coherence volume is given by the coherence length. Just envision you have a laser beam here.

And the coherence volume where all of the photons are coherent is the area of the laser beam times the coherence length. And whenever you find a photon in this kind of volume, then it is in a Heisenberg Uncertainty limited or Fourier-limited state. It is in, if it's a wave packet and pulse laser, it's in a time dependent quantum state. But it is as coherent. It is fully coherent.

So you have to figure out for your system what is this minimum-- what is the coherence volume, what defines the fundamental phase space cell of your system? And now you do the following. You are asking, what is the probability to find two particles? And what is the probability to find one particle?

So if p is the probability to find one particle, then you have three options. p square, $2p$ square or 0 to find two particles. One is the classical case of distinguishable particle.

This is the case of bosons, bosonic atoms or photons. Collins Therefore includes the Hanbury Brown-Twiss experiment. And 0 is fermions.

And the question is now different experiments, how do you define, with spatial resolution or temporal resolution, the coherence volume? And that can involve transverse coordination, temporal resolution, and whatever you can use to define the phase space cell or the coherence links, the coherence volume of a laser beam.

Let me give you one example, which I think illustrates it. And this is the example of an atom cloud. If you have an atom cloud, all the particles which are in a volume of a thermal de Broglie wavelengths are coherent.

The momentum uncertainty of particles in a thermal cloud is 1 over the thermal-- is the thermal momentum. And according to Heisenberg, the position uncertainty related with this momentum spread is just the thermal de Broglie wavelengths.

So you can say that all the atoms in a cubic de Broglie wavelengths are coherent. All the atoms are in one semi-classical quantum state. So therefore, this is sort of atoms in one single mode. So the Hanbury Brown-Twiss experiment with atoms, or the measurement of the g_2 function, could be defined as follows. You have an atom cloud.

And if you had an electron microscope or some high resolution device, you grab into your cloud and ask, what is the probability for one particle. What is the probability for two particles? And you will find that p_2 is 2 times p_1 squared. If your volume is too big, you lose the factor of 2, because you average over uncorrelated volumes.

Now, in your homework, you were looking at the question, how can I really grab into a cloud and just pick out atoms out of one phase space cell of effective size, λ de Broglie cubed? And the way how it was done is that you take an atom cloud and you drop it and expand it. When the cloud expands, there is a mapping from momentum space into position space. Then you use some form of pinhole, which provides transverse coordination.

You use a detection laser, which gives you temporal resolution. And well, this was part of your homework, but I just wanted to give you the bird's view on it. By controlling the transverse correlation and the temporal resolution of the detection, you create a situation that what you count only atoms which originated from one phase space cell in your cloud.

So in other words, let's say experiment where without electron microscope, without submicron spatial resolution, you can literally grab into a cloud, capture a volume of the thermal de Broglie wavelengths, open your hand, and figure out how is the probability for two particles related to the probability of finding one particle. Any questions?

All right. OK, good. So it's time for new chapter.

So in this chapter, we want to look at interactions between light and atoms using Feynman diagrams. And I know this part of the course is a little bit formal. We are

using exact solutions for the time evolution operator in quantum physics. But I'm not doing it to teach you sophisticated formalism. Well, it's interesting to learn it anyway.

But it is this picture of really writing down the exact solution of the time evolution operator, which helped me to understand much better what virtual states are, what certain virtual photons are. So there are things we talk about it all the time. And the question is, how do you define a virtual state? What is the virtual emission of a photon?

And the only way how I can really explain it to you is by showing you the equation and say look, the virtual photon is just a term in this equation. So that's the goal of this chapter. I don't want to overemphasize the mathematical rigor, but I want to really show you what it means to have virtual photons and what exactly virtual states are.

So let me first motivate that by reminding you of two diagrams. In physics, we always like to draw something-- a few lines, a few doodles. And when we have a two level system, we have the two processes which are emission and absorption.

Well, so this is easy as long as the light is in resonance. But if the light is not in resonance, we may ask what about this process? And since somehow the wavy line of the photon ends here, I may even put in a dashed line and say, this is a virtual state. What does it mean?

Or we can even ask, is it possible-- I'm just playing with straight lines which are quantum state and wiggly lines-- is it possible that this process happens. That would actually mean that an atom in the ground state, just out of the blue, emits a photon. Can that happen?

And I can again draw a dashed line and say, here's a virtual state. Or, we can say we start in the excited state. The atom is in the excited state. But can the atom-- its a two level system, so we don't include higher states-- but can a two level atom in the excited state absorb another photon?

So let me just change the color of these photons to red, because I want to-- and this

would remain black. All right.

Good. So question is, are those processes possible? Yes or no?

AUDIENCE: Yes.

PROFESSOR: Yes, they are. And they have experimental, observable consequences. However, they look funny because something seems to be strange with energy conservation. And what we will see is that in the end, at the end of the day, that means if you let the system evolve for a long time, after a long time, energy has to be conserved. And therefore, we will always need a second photon to conserve energy.

So those weirder processes, where photons are emitted by ground states out of the blue or excited states absorb other photons without-- also, there is no higher lying state. That looks a little bit strange.

So what you will see is-- and I want to show you that mathematically-- if an atom emits a photon out of the ground state, you would say, where does the energy come from. But quantum mechanic allows us to violate energy conservation by an amount ΔE for a time which is $\hbar / \Delta E$. So for short movement, energy can be violated.

But then, you need the second photon to reconcile energy conservation. You can violate energy only for short times. And if you would say, what does it really mean to violate energy conservation for short times? Well, I want to show you the equation, which exactly explains what it means.

Next question, just to see. If you have an excited state, and you just say it's possible to absorb another atom, you are in the excited state, and you absorb another photon. And at least you said, yes this is possible. My question is, when you are in the excited state and absorb a photon, in which quantum state is the atom after absorbing a photon?

AUDIENCE: Ground state.

PROFESSOR: Pardon?

AUDIENCE: Ground state.

PROFESSOR: In the ground state. Yes. There is no other state. And the operator-- and I want to show you that-- when photons are exchanged, always transforms the ground to the excited state, because a dipole operator connects the ground to the excited state.

So therefore-- but we'll see that all in the time evolution operator in the formal solution that this dash line is actually the ground state. And while it's not obvious, what is this dash line?

AUDIENCE: The excited state.

PROFESSOR: It is the excited state. The atom in the ground state emits a photon. Therefore, the atomic system has now lower energy, because a photon has been emitted. But the character of this state is now the excited state.

So we will later see that this means that in a perturbation analysis, we are violating energy. We are violating energy, because the real excited state is ω photon plus ω atom, the resonant energy for the atom. The real excited state is ω photon plus ω atom higher. So you will actually find that in a perturbation analysis, this term has an energy denominator, which is of resonant by exactly this separation.

And this term, because the real count state is here, the virtual state here which is the count state is here, has actually the same energy denominator as that state. And for this state, of course, this is something you've seen of resonant light scattering. The energy [ω denominator ω] is there.

So, OK. So I'll explain to you that we involve virtual states. So when we have virtual states, we want to clarify what is their energy. But we discussed that already in the discussion. And what is the wave function, ground or excited state. Yes.

So just to repeat it, when we draw those diagrams, those diagrams show the energy of the atom, but taking into account the energy of the photon. In other words, if you

have a ground state, and we absorb a photon, the photon has disappeared, and the energy of the atom is now the ground state and the photon energy. And this is a dashed line.

Similarly, when an atom emits a photon in the ground state, the photon is emitted. Therefore, the atomic energy is lower than the ground state by the photon energy. And we draw the dashed line here. This is what we mean when we say that is the energy of the virtual state. And this is what we draw in those diagrams.

By the way, yesterday we had the CUA seminar. And the speaker was actually talking about trapping atoms with quantum fluctuations. And he conceded explicitly Casimir forces, forces of the vacuum. And diagrammatically, forces of the vacuum come, because the ground state atom emits a virtual photon and reabsorbs.

This is actually, I will mention it later, the same diagram is actually also the diagram which leads to the Lamb shift. These are all sort of when you have atoms in the lowest state, and they interact with the electromagnetic field. Well, if they interact with the vacuum, they cannot absorb photon. All we can do is emit a photon, and this leads to this diagram.

But just to tell you that you have to be careful, yesterday's speakers actually used this process. But he drew this diagram. He drew it in the opposite way, which I think confused some people in the audience. The correct way is to draw it like this. The virtual state for vacuum fluctuations for the Lamb shift for the Casimir force is below the current state.

OK, so we have-- let me just redraw those three diagrams. I want to now introduce the time evolution. So I want to use those. Seems I'm running out of space.

So we want to introduce the time axis for-- so we had this diagram. We had this diagram. And we had-- this is a virtual state. OK.

So those three diagrams correspond to the following situation. If time evolves from the bottom to the top, we can now draw the first process when atom in the ground

state first absorbs and then emits a photon. And let's label the second photon in the purple color.

So an atom starts out in the ground state. Then it reaches the time where it interacts the electromagnetic field. There is a photon which propagates from earlier times to this time, t equals t prime. At the time t prime, the photon disappears.

And the atom, which was in the ground state, goes to the excited state. It may propagate in the excited state for a while. And then it emits the photon and is back into the ground state.

So this is the temporal diagram for this process. For the next situation, we also start with an atom in the ground state. But now we have the situation that a photon is emitted by the ground state. As a result, the atom switches to the excited state. Strong violation of energy, but possible for short times.

Then there is a real photon, which is now absorbed, absorbed by an atom in the excited state. And that takes us back down to the ground state. And finally, in the third scenario, we have an atom in the excited state.

There is a real photon which is absorbed that switches the atom into the ground state. And then the atom in the ground state emits a photon. And as a result, it is back in the excited state.

So let me just make a note that when a ground state atom absorbs a photon, this is the co-rotating term in the quantum description, which a lot of you have seen. The opposite process of emitting a virtual photon-- I shouldn't say virtual photon. It's a photon. What makes it virtual we will see later when we have the more accurate mathematical formulation. So this is a counter-rotating term.

And we have used the rule that every photon, which with a real or virtual-- every photon which is interchanged with the atom absorbs or emitted-- changes the atomic state from ground to excited. So an atom goes from the ground to the excited state either by photon absorption or by photon emission. Both is possible.

So whenever photon appears or disappears, it changes-- let's say we start in the ground state. We go to the excited state. And this is possible by photon absorption-- this is a co-rotating term-- or emission.

What I've just said reflects that when we derived the dipole approximation, that the essential term of the dipole operator is off diagonal. The dipole operator is an operator between calm and excited state. Therefore, it has those two matrix elements.

Whereas electric field operator is the equation and annihilation operator a and a dagger. Or which is very elegant if I use for the two level system a spin one half description. And I use sigma plus and sigma minus, raising and lowering operator.

Sigma plus takes the ground state to the excited state. Sigma minus takes the exciting state to the ground state. Then in the fundamental atom light interaction, we have those four terms.

And now you see that sigma plus takes the atom from the calm to the excited state. But sigma plus appears both with photon absorption and with photon emission. So therefore, the atom can-- this is what the quantum mechanical operator tells us-- go from the ground to the excited state either by photon absorption or by photon emission. Any questions? Yes.

AUDIENCE: So if we were to look beyond the dipole approximation, would we see transitions that leave the [INAUDIBLE] state?

PROFESSOR: Yes or no. What we need here is a bilinear interaction, which has an raising and lowering operator for the atom. And here it has a raising and lowering atom operator for the photon number.

And it has the co-rotating term where you raise here and you lower there. You lower here and you raise there. But it has also terms where you raise the atomic excitation, and you raise the photon number by one, which of course violates energy. But the message I want to give you is this is possible for short times. And this is exactly what the quantum mechanical equations tell us.

So the short answer to your question is, as long as we describe the system by the dipole interaction, or what we have even beyond-- I'm not using a perturbation argument. I'm writing down the operator. And this operator can now effect the system to all possible orders. So even the non-perturbative strong coupling limit will always have products of atomic raising operators, lowering operators with photon raising and lowering operators.

The little bit longer answer is, some of what I'm telling you may be gauge dependence. If you work in the-- if you use, not the dipole description, but you use the $p \cdot a$ Hamiltonian, this Hamiltonian has a e^2 term. The e^2 term allows the atoms to exchange two photons at the same time.

So in other words, in the a^2 interaction, we may have a vertex point in time where the atom interact with the photon field where two photons are exchanged. And then the rules are different. So in that sense-- but this is quite often when we give a quantum mechanical pictures, which has terms, which nicely shows the time evolution.

And you think wow, yeah, this is what happens. One photon at a time, the atom goes ground, excited, excited, ground state. And you have your picture. And it is fully consistent with the exact quantum mechanical result. You may be able to get the same result out of a different picture.

That point will actually occur again when we talk, for instance, when we talk about density operators. A density operator is an average of our quantum states. But you can get the same density operator by averaging quantum states in different basis sets. So if you get one picture out of one specific basis set, it gives you the correct intuition, it's a correct description, but it is not the only possible description. [? Collin ?].

AUDIENCE: Maybe the thing, I don't know.

AUDIENCE: I think what you're trying to get at is that picture works as long as you can limit yourself to a two level system. Because that's correct on a whole. Or is it like strong

coupling? But as long as you can identify the two level system, or I think the dipole versus particle approximations [INAUDIBLE] an atom may have transition probabilities to others, like from an [INAUDIBLE] state. But--

AUDIENCE: Actually, I was just wondering if it was possible to have it like a transition matrix element to [INAUDIBLE] coupling the ground state to the ground state.

PROFESSOR: Yes. The e^2 operator does that, because in the other gauge, which is pa , pa is in dipole operator. p is the momentum operator for the atoms. And the p operator will switch the atom from ground to excited state. a , the vector potential, involves a and a^\dagger .

So the $p \cdot a$ term does pretty much the same as the $d \cdot e$ term, because it's a bilinear product of an operator which has an effect on the atoms and has an effect on the photons. And it means you always switch the quantum state in your atom and create or annihilate the photon. But the a^2 term-- it's just a square, it doesn't have a p in front of it-- will allow an atom in without changing the state to create two photons, emit two photons. And then there's probably a cross term which means to absorb and emit a photon.

You will actually, in one of your next homework assignments, you will actually look at the a^2 term. And I think it's a very educational problem. So you really see that you have two very different descriptions, but the results fully agree.

But this is already an expert discussion. Why don't we for now just hang onto the dipole approximation. We have $d \cdot e$, and we just learn what is inside the dipole approximation, and what is an exact quantum mechanical description using the dipole approximation.

But and then we can come back to the discussion, is everything we describe really real. And while the answer is, what is real in quantum physics is the final result, the intermediate results, you have to say that this is only one possibility to go to the final result.

OK, so we want to-- let me just put in one more page. So what we want to do now is we want to do a calculation. It's a perturbative calculation, but we can take it to all orders. So therefore it's exact and general. We want to do a perturbative calculation of transition amplitudes.

And what I am following here the discussion in atom photon interaction on those pages. And the idea is the following. We have the Hamiltonian for the atom. And then we have a Hamiltonian which describes the transverse field. And the interaction of the transverse field with the atoms.

What we want to use is we want to use the interaction picture, which is often the nicest picture to describe the effect of the interaction between two systems. In the interaction picture, you are transforming from your Schrodinger type ψ function to wave function ψ tilde. And those wave function ψ tilde include already the dynamics of the Hamiltonian H_0 .

So that means that all operators in the normal Schrodinger picture become now operators in the interaction picture by canonical transformation, which involves the dynamics due to the unperturbed Hamiltonian. And so that's just a reminder what the interaction picture is. And we are now interested what happens in the interaction picture to an initial wave function ψ of t_i .

This wave function already includes the time evolution due to H_0 . So the only change now comes because we have interactions with the electromagnetic field. So this allows us now to focus only on the relevant interactions, the interactions we want to understand.

And the formalism which is used in atom photon interaction focuses on the time evolution operator, which I'm sure all of you have seen. So the time evolution of our system, how the wave function evolves from the initial to the final time, is described by the time evolution operator.

I will come back to the wave function later, but in order to derive the time evolution of the wave function, it's at least convenient to focus first on the time evolution

operator. So in other words, I want to show you a formal solution for the time evolution operator. And once we know the time evolution operator, we apply it to the wave function. And then we are talking again about the wave function.

OK, since the derivation involves many equations and they are all printed in the full beauty in atom photon interaction, I decided to use pre-written slides here. Also, most of it is fairly elementary. The equations are complicated, but the concepts behind them are very, very simple. It's actually the beauty of using a kind of a time evolution operator. It's a little bit mathematically formal, but it allows us to express what is really going on in the solution in very, very simple terms.

So the goal is to do a perturbative expansion for the time evolution operator. If we have no interaction potential v , the interaction picture, nothing happens in the interaction picture, and therefore, the time evolution operator is a unity matrix. But if you have a coupling term v , there is time evolution.

And the time evolution can be treated in lowest or in higher orders in the perturbation. And we want to use now perturbation expansion, where we have correction terms, which is the first order, second order, and n th order correction for time evolution operator. So what we simply want to do is we want to do an iterative solution. We want to find an iterative solution for the time evolution operator.

The next two lines is just a reminder. What is the differential equation for the time evolution operator? Well, we are really just talking about Schrodinger's equation. Schrodinger's equation for the wave function turns into differential equation for the time evolution operator.

Schrodinger's equation in the interaction picture says that the time derivative of the wave function in the interaction picture has nothing to do H naught. H naught has already been absorbed in the definition of the wave function. So the time evolution in the interaction picture only comes from the interaction term.

And now, if you write the wave function at time t_f as the time evolved wave function from the initial time to the final time, then you take the derivative here. You actually

take a derivative of the time evolution operator. And therefore, in one step, from Schrodinger's equation, you find a differential equation for the time evolution operator. In other words, this is now the operator equation for the time evolution operator. And it's nothing else than a 100% rewrite of the Schrodinger equation.

Interrupt me if you have questions. This part should be-- we'll go through that for solely pedagogical reasons. And that means you should really understand it.

So this equation can now be formally solved in the following way. Take this expression for the time evolution operator and insert it in this equation, and you find an identity. Of course, you haven't really solved it, because you have the time evolution operator expressed by the time evolution operator. This looks like a circular conclusion, which is just nonsense.

But if you inspect it more closely, you observe that the time evolution operator here is expressed by the time evolution operator multiplied with v . And if you now think in an iterative solution that we want to express. We want to expand the time evolution operator in the different orders in the interaction parameter v . Then you find actually that if you are only interested in the first order of the interaction operator, you can use the [INAUDIBLE] order here, because you get one more power of v .

So that's how you get the first order. If you want to know the second order of the time evolution operator, you can plug-in the first order on the right hand side. So therefore, you get an iterative solution.

The first order solution is one. The [INAUDIBLE] order solution is the unity operator. The first order solution is by just putting the unity operator here. The second order solution is by taking the first order solution, plugging it in in here. And then you get two integrals over the operator v . So in that sense, we have formally solved the time evolution of the system in all orders. So this is what we get out of it.

Yes, the equations are getting longer. But the structure is fairly obvious. So you saw that the-- let me just flashback. The first order term had a temporal integral over v . The second order term has two temporal integrals over v . And there's a time order

in between τ_1 and τ_2 . And the n th order term now involves n temporal integrals where the times are ordered in such a way.

Now we want to go back to the Schrodinger equation. So we want to use this already fairly complicated expression for the time evolution operator and apply it to the initial wave function. And we want to calculate matrix elements between the initial wave function and some other ϕ_f , some other wave function in a given basis.

So we are also going back. I don't want to go through too many steps. Here, we're also going back to the original Schrodinger picture. So I've taken all the tildes off.

But you know the step from the Schrodinger picture to the interaction picture was only taking out the Hamiltonian H_0 . And the Hamiltonian H_0 for eigenstates ϕ_i and ϕ_f , is simply a time evolution, $e^{-iE_i t}$ with a certain time. So all what this transformation to the interaction picture gave us, it eliminated, at least for a short while, all of those phase factors, which are simply the evolution of the eigenfunction in the Schrodinger picture.

So let me look at a typical term which we have right now. And you can't expect something simpler than this, because this is the general solution. It's an exact solution to all orders. It's a summation over terms to all orders. And eventually the summation of an infinite number of diagrams is an exact solution.

So what we've got here is we have an integral over n times. And the n times are time ordered. This is the formal solution for the time evolution operator. And that means that we integrate over all times, but what it is under the integral sign is a product of the operator V at times $\tau_1, \tau_2, \tau_3, \tau_4$.

Since we got back to the Schrodinger picture, what happens is between times τ_1 , and τ_2 , the particle simply evolves with the phase factor given by the Hamiltonian H_0 . So the picture is the following.

The time evolution operator in the interaction picture is a product over the operator V at different times. If you simply plug it back into the above equation-- and you

have to look at it for a little while to see that everything works out correctly. But what that means is then that if you go to the Schrodinger picture and write down the operator in the Schrodinger in the basis of unperturbed eigenfunction, that you have products of the operator v at times t_1 , τ_2 , τ_3 , they're time ordered. And because we have left the interaction picture, you also get the propagation with the phase factors given by the eigenenergies of the unperturbed Hamiltonian H_0 .

So therefore, you get just an infinite number of terms which have all the following structure, that the particle propagates in its eigenstate. It reaches a vertex where the interaction switches the particle from one state to the next. Then in the next state, which may now be the excited state, we have propagation in the excited state. Then the excited state is, again, exchanging a photon, has an interaction term, and is switched to-- well, for the two level system, it has to go back to the ground state. In a more general situation, it goes to another quantum state.

So therefore, what we have exactly derived is that the propagation, or the time evolution, of the wave function is nothing else than many, many of those factors. And each of those factors can be represented by a diagram like this.

So let me just write down what this diagram here means mathematically. It means that initially, we have a photon of energy ϵ , of polarization ϵ , wave factor k , and a certain energy. And we have particles in state a . That means that we get from the initial time to the time τ_1 . And τ_1 is the time of the first vertex of the first photon exchange, that the wave function evolves.

And it evolves with an energy of the unperturbed Hamiltonian, which is the atomic energy. And we have one photon $\hbar\omega$. At the time τ_1 -- and this is when the diagram on the left hand side has a vertex-- we are now bringing in the interaction operator, which acts on the atom. It takes the atom from state A to state B . And it changes the state of the photon. In this case, a photon with a certain wave factor and polarization simply disappears.

So this is now the time τ_1 . The next Vertex is reached at time τ_2 . And between the time τ_2 and τ_1 , the system evolves according to the Hamiltonian H_0 .

And the energy is now, because we have no photon, is simply the energy of the atom in state B.

Then we reach the next vertex. We start with a state B and no photon. The interaction switches, as now I assumed we have more than two levels. Two are state C. And we have now a photon, which may be different, k prime ϵ prime.

Since we are now, for this given example, I've selected n equals 2, something which is second order in the perturbation. But that means now that between the time τ_2 and the final tie-in for the time propagation, that's now everything is done. Things propagate. And the unperturbed Hamiltonian H_0 gives us now a phase factor, which reflects the energy of the atomic state, and the energy of the photon in mode k prime ϵ prime with energy $\omega \hbar$ per ω prime.

So in other words, I hope you see that at the end of this-- I mean, this is why people have Feynman diagrams. It's a complicated, involved mathematical formulation with multiple integrations. But at the end of the day, what we have derived is that the most general solutions to all orders in the perturbation is nothing else than a sum and integral over diagrams like this.

So when I depict it here, the second order diagram, of course this has an exact mathematical meaning. And what we have to do is when we solve a quantum mechanical equation, we have to now allow those interactions, which happen at time τ_1 and τ_2 , to happen at arbitrary times. So therefore, we have to sum over amplitudes by integrating over times τ_1 and τ_2 .

So on Friday-- I have to stop now. I think there's a seminar right now. On Friday, I will show you that with this description, we have actually captured everything I explained to you earlier-- those virtual states, the emission of virtual or real photons. So everything that which was maybe qualitative at the beginning of this section has now a precise, mathematical meaning. But that's what we do on Friday.