PROFESSOR: OK. First, welcome everybody and good afternoon. So the question is about the master equation. We know is that the most general master equation has to have the Lindblad form. And the Lindblad form has a jump operator which is providing dissipation. In its simplest case, a jump operator is just the sigma minus operator for spontaneous emission, which takes us from the excited state to the current state.

And when we have the Linblad form, the jump operator and it's commission conjugate, like sigma plus, sigma minus, appear on the right-hand side of the master equation with the statistical operator here, here, and in the middle. There are plus-minus signs in factors of two.

So I don't have a intuitive explanation where the factors come from. They come from the derivation. But you have a question about it, Nancy?

AUDIENCE: So after writing the Linblad form, we get gamma [INAUDIBLE], which we are saying for the r dot vector, we were writing [INAUDIBLE]. And we said that they are directly coming from the first one?

PROFESSOR: Yes. So let's go right here. That's actually what we were going to discuss today.

We want to discuss, today, solutions of the optical Bloch equations. And just to remind you, we derived a master equation for the density matrix. And this master equation, through the interaction with the reservoir, is acting a term to the Hamiltonian evolution. And this is showing right here.

These are the interactions with the reservoir. And in the case of the optical Bloch equation and spontaneous emission, we have only one Lindblad operator, which is Lindblad operator, sigma minus, for spontaneous emission. So this here is the master equation for the two-level system interacting with a vacuum through spontaneous emission.

And by substituting the two-level density matrix, has three non-trivial components-- sure, it's a 2 by 2 matrix, but the craziest one-- and so we can transform from the matrix elements of the density matrix to three other coefficient, which come in very handy, because they allow a simple geometric interpretation. And this is the Bloch vector.

And so the master equation, which is shown here, turns into a differential equation for the Bloch vector. And those two equations are identical, we've just done those substitution. But your question is now about--

AUDIENCE: Plus gamma, because we have negative gamma over 2, which is from the first and last term. But the gamma term should be positive, right?

PROFESSOR: This one here?

AUDIENCE: No.

PROFESSOR: This one?

AUDIENCE: Yes. Because that's negative gamma over 2 times negative 2.

PROFESSOR: No. I mean, first of all, this is a differential equation for r dot. OK. This is a differential equation for the Bloch vector. And we have here three different relaxation rates, which are gamma, gamma over 2, and gamma over 2. And they are all negative, because they are relaxation rates.

This here is not a perfecter of r. It is sort of a constant. And it means, in the long time limit-- I mean, what happens if you have a system which spontaneously decay after a while? The r vector is not 0. The r vector is hanging down. The system is in its ground state. And this is exactly that.

Wait. Am I r dot? Yes. The effect is the system, the differential equation, has to relax towards the equilibrium. You can actually re-write this equation, r dot equals a matrix times r minus r equilibrium. So here is the matrix of the optical Bloch equation. So if you re-write that that r dot is the matrix times r minus r equilibrium, then r equilibrium times the matrix gives something constant. And this is exactly this here.

But it's a mathematical identity. Just look at it. There is no assumption. It's an exact re-writing of the equation. So it is this equation which will be in the focus of our attention not only today, but also when we talk about light forces.

The first thing we did is we wanted to discuss the Mollow triplet. And just as a reminder, we have talked about, at the beginning of last class, we talked about the fluorescence. An atom is excited by mono-chromatic laser light.

And at low intensity, a delta function emits exactly the same frequency of light which is provided by the laser light. So the incoming and outgoing photons have exactly the same frequency. And this is simply a consequence of energy conservation.

But at higher intensity, we have sidebands. And we understand intuitively that we have sidebands because the system is Rabi oscillation. So same, classically, we have an emitter which has some modulation that creates sidebands.

But is rather easy to understand is, by diagonalizing the 2 by 2 matrix, we find that we have a splitting of excited state with n photons and grounds it with n plus 1 photons. So we have sort of a plus and minus state. And if we look at the possible transitions, we immediately find the explanation for the Mollow triplet. There are three different frequencies which can be emitted in transitions between rest states.

What is much more subtle, what cannot be obtained from a preterbative treatment, is the width, how wide are those peaks in the Mollow triplet. And there is tens and tens of pages in atom photo interaction. But at least, for two limiting cases, I could show you what the width of the peak is by first saying, if you have delta equals 0 and g equals 0, the matrix, which determines the dynamics, has three eigenvalues, gamma over 2, gamma over 2, and gamma.

And now, if we detune, we add a rotation along z. If you drive the system strong, it corresponds to rotation around x. You probably remember, from 8.421 the spin in a magnetic field. It rotates.

In a frame, rotating with omega, it rotates at the detuning delta. But if you drive it, we make it flip along the x-axis. So we find exactly that here.

And then, it's an intuitive way to sort of matrix, if you want, solve the matrix mathematically. But intuitively, what happens is, if you have a z-rotation, we don't change in this eigenvalue. And the rotation is just aiding i omega-- omega is the rotation-- to x and y, because x and y are now rotating.

So therefore, we obtained that, in the case of detuning the z-rotation, we have minus gamma, minus gamma over 2, minus gamma over 2. And here, what appears here is the rotation frequency. And then, if you rotate around the x-axis, then, of course, nothing happens, but what used to be the x-axis. So we get one eigenvalue with minus gamma over 2. But the y and z eigenvector are rotated.

And hand-wavingly, I can tell you. But you can show it mathematically that this means the two new eigenvectors have the average of those two damping terms. And this is 3/4 gamma. So at least, for the two cases for large detuning and strong drive, I was, with a little bit of intuition, showing you and deriving for you where the non-trivial bits of the Mollow triplet comes from.

So I hope you're, at least, impressed that these optical Bloch equations are powerful and allow us to make predictions which would be difficult to obtain otherwise. Any questions about that? About the Bloch vector, the differential equation, and the spectrum of the immediate light?

OK, we want to now discuss one other aspect, which is actually fairly simple. We want to discuss steady state solution. For the steady state solution, we just go to the differential equation and say, the left-hand side is 0. That immediately gives us a solution for the Bloch vector.

It gives us a Lorentzian. But for strong drive, it has a term here, which we will immediately realize that this gives us power broadening. And then the steady state

Bloch vector is g delta, g gamma over 2, delta square plus gamma square over 4. So we can now use this solution for steady state and discuss everything we want what happens in steady state.

So one question is, if you have an atom laser beam in steady state, how much light is absorbed? Well, the basic equation is, if you have a charged q which moves in an electric field, that means work is done. So therefore, the absorbed power is related to that.

And we can now combine q dr/dt. And this is nothing else than the derivative of the dipole moment. And then we calculate the average.

So with that, we find the result they the absorbed power is given by-- just one second-- e0? Yeah. We express now e0 by the Rabi frequency. And h bar appears in omega.

But the important thing is that we need now the dipole moment or the derivative of the dipole moment. And the dipole moment is given by the coherences of the density matrix. And that is, if you look at the substitution, is the x and y component of the optical Bloch vector.

But now what we need, of course, we need only the component of the dipole moment, which gives a non-vanishing term with cosine omega t. That means, what we need is-- I mean, that's like an harmonic oscillator. The component of the motion which is responsible for absorption is the one which is in quadrature, which is assigned only the d-component.

And that, you have to go back and look by inspection. But this is exactly what the y-component of the Bloch vector does for us. It will become important later on for optical forces.

So we have the optical Bloch vector. We are in the rotating frame. And the one component, which is the x-component, is in phase with the light. And the y-phase is 90 degree out of phases in quadrature. And for absorption it's, of course, the in quadrature component which is relevant and which determines light scattering.

OK, this is the absorbed light. And we don't measure power in watt, we measure power in the number of absorbed photons. This is the natural unit here. And for that, we have to divide the expression by h bar omega. And that means now that the number of photons absorbed per unit time is given by the Rabi frequency and the y-component of the Bloch vector.

Well, we can ask another question and ask, what is the number of emitted photons? Well, the number of emitted photons is related to the population in the excited state. We take the population in the excited state and multiply with gamma. This is spontaneous emission out of the excited state.

Now, the excited state can be-- the excited state population, you have to look up at the substitution, but is related to the z-component of the optical Bloch vector. To remind you, the z-component is the difference of population between excited and ground state. But the sum of the two is one. So therefore, this is now the excited state population.

And here is our steady state solution. Here is our steady state solution for the z-component. And by using that now, we find that we have the result with the Lorentzian denominator. And what we have here is gamma g square over 4.

And now, of course, if you compare that with the solution for our y, for the number of absorbed photons, you find-- well, it shouldn't come as a big surprise-- that the number of absorbed photon in steady state is equal to the number of emitted photon. So this is one case, one important part of the solution.

And I should have shown that to you earlier. When we look at the two components, the x and y-component, the y-component is absorbed, if it is Lorentzian. Whereas, the x-component is dispersive.

Just by inspection if these two expressions, here is a delta. And the delta makes the x-component anti-symmetric. And the y-component is symmetric with respect to the origin.

So what we see here is we have a Lorentzian. The Lorentzian has a natural line broadening by gamma. So the full width at half maximum is gamma, but only in the limit of 0 drive. When we drive it, we have an additional broadening, which is called power broadening.

So for absorption, the full width at half maximum is 2 times gamma square over 4 plus g square over 2. And this last term goes by the name power broadening or saturation broadening. Just to avoid common confusion, if you go to very high power, that least term doesn't play a role.

How does the line width scale with power in power broadening? Linear in power? Quadratic in power? Square root of power?

AUDIENCE: Square root?

PROFESSOR: Square root of power. Yeah. So in that sense, yes, power broadening only goes to the square root of power. It's, in essence, the Rabi frequency.

If you drive the system at the Rabi frequency, it gets power broadened by the Rabi frequency. And I think that's a pre-factor which is 2 or square root 2. But in essence, power broadening scales with the Rabi frequency.

So let us quickly discuss the case when we go to very high power, very high Rabi frequency. At that moment, you can just see it in the solution on the page above. The z-component of the optical Bloch vector becomes 0. That means there is no population difference between excited and ground state anymore.

We often describe the limit of high power by introducing a saturation parameter, which is defined here. Saturation is discussed in more details in Part 1 of the course. But I want to relate it here to the solution with the optical Bloch vector.

The z-component, which is the difference between ground and excited state population, now has a very simple form, 1 over 1 plus s. The population to be in the excited state is 1/2. That's what we get at infinite power. And then, if the power is finite, it is multiplied by s times s plus 1. In the power broadened line width is the natural line width, which gives the line width in the limit of low power, and then you multiply with the square root 1 plus s.

So what does it mean to have a saturation parameter of 1? Well, if you just look how it was defined, if you have the ground and the excited state at a saturation parameter of-- first, at a saturation of infinite, you have 50-50 population. And at a saturation parameter of 1, you are 1/2 way to 50-50. That means you have 3/4 population here and 1/4 population there.

So at this case, the number of photons emitted per unit time, which is always given by this formula, becomes now gamma over 4. So at a saturation parameter of 1, the emission rate is gamma over 4. At a saturation parameter of infinity, the spontaneous emission rate is gamma over 2.

Any questions about steady state solution? [INAUDIBLE]?

AUDIENCE: So what we derived right now is a spontaneous emission rate. Does the optical Bloch equation say anything about the stimulated emission? Or is that the fact that it just doesn't have [INAUDIBLE]?

PROFESSOR: Stimulated emission is built in the optical Bloch equations for negligible damping contained Rabi oscillation. And Rabi oscillation is the stimulated emission. So this is being built in.

However, it seems to be a describing the drive field by a classical field by a Rabi frequency. We are not really accounting for the number of photons exchanged with that. So in a way, the driving field is always treated in the undepleted approximation. It's a C number.

But in the drive through system, but of course, you can immediately lead from the Rabi oscillation the exchange of excitations between the atomic system and the drive field. So yes, the optical Bloch equations contains that, but not in the photon picture.

AUDIENCE: If you have Rabi oscillations, you can kind of guess that, every time you go down you simulated the events. In space, say, you don't have Rabi oscillations. Can you still somehow extract the rate?

PROFESSOR: I hope the question we answer will become crystal when we discuss quantum Monte Carlo simulations.

AUDIENCE: OK.

PROFESSOR: In quantum Monte Carlo simulation, we have an ensemble of atom. And the atoms all do Rabi oscillations. But in steady state, they do it out of phase.

So you have atoms in your ensemble which, at a given time, absorb. And others emit in a stimulated way. And the effect cancels out in steady state. Other questions? Yes?

AUDIENCE: So you're saying, in steady state, they kind of de-cohere. And you don't have any net change in those populations. Do you still get the triplet lines if you don't have a modulation of your intensity beam?

PROFESSOR: Yes. Absolutely. The Mollow triplet is a feature in steady state. And maybe let me explain that, because it's an interesting discussion.

What will happen is, in steady state in your ensemble, you don't have Rabi oscillations. But if you would look at one atom, you see Rabi oscillations. It's just when you average over all atoms in your ensemble, the phase of the Rabi oscillation has averaged out.

So in other words, for instance, what would happen is the following. If you take a steady state solution and you ask, what is the dipole moment as a function of time in steady state, you will find that this is 0. So you would then ask, hey, where is the light emitted, because isn't the light emitted by a time-dependent dipole moment?

What you really have to do is, when you want to study light emission in steady state, you can't look at expectation values because they're not changing as a function of time. What you have to do is you have to look at the correlation function.

And the correlation function sort of tells you-- let me just use it for Rabi oscillation. If one atom is in the excited state, a Rabi period later, it would be again in the excited state. So although you are in steady state, you have correlations because, if you look at your ensemble and see there is a fluctuation where the atom is excited, you will actually find a co-related fluctuation a Rabi period later.

So in steady state, when you want to study dynamics like the Mollow triplet, you should not look at the solution for the dipole. You should look for the solution of the correlation of the correlation function. And I made this cryptic remark on Wednesday that this correlation function follows the same differential equation as the expectation value. Therefore, when I try to discuss for you the line width of the Mollow triplet, I simply use the matrix for the Bloch vector knowing that it is the same matrix, the same differential equation which will describe this correlation function.

But it is important here that, in steady state, the average values are 0, but we have fluctuations. And it is the fluctuations which emit the light, the fluctuations which emit the Mollow triplet. And fluctuations are described by quantities like this. So it's a little bit beyond what I want to discuss in the book, but atom photon indirection has a wonderful chapter on that. Other questions?

There is another important case of the optical Bloch equation, which is the weak excitation limit, which is simple. But I've already asked you to look at it in your homework assignment. OK, let's now come to the third aspect of master equation solutions of optical Bloch or master equation. And these are damped vacuum Rabi oscillations. Yes.

I really like this example in this chapter, so I hope-- well, I think we have many highlights, but this is a really nice example. I learned it from Professor [? Schwan ?] when he introduced it. And what I like about it is I can use it to show you what other Lindblad operators may be important. So you suddenly understand, in a bigger context, what the master equation is.

And then we continue with an atomic cavity. And the damping is no longer by spontaneous emission. The damping is by photons sneaking out of the cavity. So we really learn something which is similar to optical Bloch equation, but in another context. And often, if you see two different realizations of similar physics, you maybe realize more what is more generic and what is special for instance of spontaneous emission, so I really like that.

But also, I found that this example allows me to introduce to you the concept of the quantum Zeno effect and the concept of atypical elimination of coherences. So it's a wonderful example which connects us with a number of really neat concepts.

So with that promise, let me remind you that, in the master equation, when we derived it, we got an exact expression when we did second order perturbation theory. And the structure which came was this double commutator between the interaction operator and the steady state operator.

Just to remind you, in the interaction picture-- first in the non-interacting, in the normal picture, the derivative of the density matrix is a commutator with h. In the interaction picture, it's a commutator with v with the interaction. But when we iterate an equation to second order, we plot the first result into the equation. Then we get, in second order, this structure.

And if I, by expanding the commutator, we obtain the following structure. And that gave rise to the Lindblad form of the master equation, which is rho dot equals a Hamiltonian part. But then we have now a sum, v, the interaction with the environment, can be a sum of mu dot b and b-- and so an external b field times magnetization and dipole moment times an electric field.

It can have a number of terms. And therefore, in general, we have more than one Lindblad operator. But the structure is always given by this commutator in the following form.

OK. So we have this general derivation. But until now, we have only looked at a system which is a simple atom. And the environment was the vacuum. So this is our environment.

And also, the environment was always in the vacuum state. So these were our assumptions. And that means that the only Lindblad operator, the only jump operator in this sum is sigma minus for spontaneous emission. And by inserting this jump operator into the Lindblad form of the master equation, we find the optical Bloch equation.

But now let's do cavity QED. And there are lots of experiments going on in the research group of Professor Vuletic. And in this situation, our system is actually not just the atoms, it's the atoms and a single mode of the cavity. This is our system. And the environment are all other modes.

So in other words, the atoms interacting with a single mode of the cavity, this is our system. And the environment is now accessed by emitting-- the atom is emitting. Or there are photons leaking out of the cavity, because the cavity mirrors do not have 100.0% reflectivity.

I will immediately reduce it to one Lindblad operator. But we could now say that this system, in its full glory, has four different Lindblad operators which provide dissipation, which provide coupling to the environment. The first one is simply what we had before, spontaneous emission.

However, if we allow that the environment is thermal, we have to multiply. Well, whenever you have emission and you have already population in a mode, and thermal is a number of thermal photons in every mode, on average, then you have an extra bosonic or photonic stimulation term. That's stimulated emission, but not by the laser beam. It's now stimulated emission by a thermal occupation of all the other modes.

So just assume, for that purpose, that you have a cavity here. But the cavity is now in a black body cavity. And everything, all modes in the vacuum, are no longer vacuum modes. They are occupied within thermal photons.

Of course, if this is the case, we have also the opposite effect. You can see by unitality, namely, that we can have-- oops, n should be-- but that we have an absorption process. We absorb a photon from the thermal background.

But now, we have two more processes. And this is photons leaking out of the cavity. When a photon leaks out of the cavity, this leakage is not coupled to the atom, it's coupled to the mode.

We describe the mode by a mode operator, a. So now, the leakage out with the rate, kappa, is presented by that. But in the case of thermal photons, we have stimulation by the thermal photons. And also, if we have thermal photons, photons cannot only leak out of the cavity, they can also leak into the cavity. So we have assumed here that we have n thermal photons in each mode of the environment.

So in other words, if you want to look at, for instance, Serge Haroche does experiment in the wonderful experiments in the microwave domain. He has cavities at, I think, gigahertz frequencies. And even at cryogenic temperature, there are occasional thermal photons in this mode. But you are now expert enough that you would know how to set up a master equation for that situation.

But I want to discuss simple limiting cases. So I think you are glad to hear that we are choosing 0 thermal photons. So therefore, we assume we have a cavity in vacuum. And therefore, we only need the Lindblad operators, L1 and L2.

And we want to describe the system which has only one x, which has, maximally, one excitation. We try to keep the density matrix small, so we only want to consider three states. So when I said there is one excitation, it is, of course, the state which has no photon in the excited state.

The state two is we have the ground state and one photon. And those two states are coupled by the cavity. And since I really want to discuss the simplest case which already illustrates all the concepts I want to do is they are coupled by the vacuum Rabi oscillation.

And we have discussed vacuum Rabi oscillation in part one of the course. But it's very simple. It's just two levels coupled. It's a 2 by 2 matrix, so this is something you are familiar with.

But now, the master equation, the Lindblad operator bring in the third level, which is the ground state without photons. And this can happen in two ways. One is spontaneous emission. The excited state emits, but not into the cavity. It emits to the side.

So spontaneous emission can take us down here, or when the excitation is in the cavity. We have one photon in the cavity, this photon can leak out. And the rate that is kappa.

And since we have talked about spontaneous emission so much with the optical Bloch equation, we want to discuss now the case where gamma can be neglected. And we want to understand what happens when the only dissipation of the system comes by photons leaking out of the cavity.

Any questions about the system and the motivation? So our system is now an atom with a cavity. The environment is a vacuum. But the process of dissipation relaxation is no longer spontaneous emission, it is the photon leaking out of the cavity.

And this realizes different physics. And I hope, by experiencing these different physics, you have sort of a nicer picture what is dissipation in an open system and what is similar, but what is also different or spontaneous emission. [? Koren? ?]

AUDIENCE: If you're considering only kappa, isn't there an equivalent process of order kappa that takes you to ground of two photons in the cavity? Don't you have to consider that state as well? That term would be of order kappa as well.

PROFESSOR: So great. The question is I restrict the Hilbert space to maximally one excitation. So this is how we start out with our differential equation.

AUDIENCE: Oh, this isn't in a thermal bath? I thought we were assuming we had a--

PROFESSOR: Oh, sorry. I first made everything complicated. But then I said, now, let's make it simple. We assumed the thermal bath is 0. So we reduced--

In a way, yes, I told you here are four Lindblad operators, and you can describe everything you want. But then I said, hey, two of them become 0, because we eliminate the thermal bath. And now I make the approximation that spontaneous emission is negligible. And now we are back to the situation which is nice for classroom discussion. We've only one term left, and this is the leaking of the photon out of the cavity.

So I will show to you what happens. But ultimately, when we make the cavity better and better and kappa smaller and smaller, I will say, wait a moment. We now have to check that our solution is still consistent with our assumption that spontaneous emission can be neglected. So that's sort of what you want to do.

Yes. Let's just get two more pages. OK.

So what we want to do is we want to look for the dynamics of the system. We start out in state one. So we inject one excited atom into the cavity. And what we want to learn now is-- and it has a lot of interesting physics in it-- what will happen as a function of this new dissipation, kappa.

Well, qualitatively, it should be pretty clear. If kappa is 0, we have vacuum Rabi oscillation between state one and state two. If you then put a little bit kappa into the system, you have a Rabi oscillation, but they are getting damped.

And like in any oscillator, if you crank up the damping, you go into an overdamped regime. And this is exactly what the equations give us, Rabi oscillation, damped Rabi oscillation, and then overdamped regime.

And actually, the most interesting regime for us will be the overdamped regime. That's something we haven't encountered. But let me just, before we go there, write down the master equation for you.

So without the photon leaking out of the cavity, we have the following terms. Without the photon leaking out of the cavity, of course, state three is not involved. We simply have Rabi oscillation between state one and state two.

And this is actually also like the optical Bloch equation. Without damping, it's simply the Jaynes-Cummings model. Minus, that's this. And for the coherences, we get omega 0 rho 1, 1 minus rho 2, 2.

But now, we put in the terms with kappa. The state, which is the state two, which is the ground state with the photon, has now a damping term, because the photon can leak out. We get, of course, an equation for the ground state, the population, the photon leaks out.

And we populate state three. And the damping also affects the coherences. And the form of the master equation, the Lindblad form, gives us the following damping term.

So there are two new concepts which I promised you to introduce. One is the idiopathic elimination of coherences. And the second one is the quantum Zeno effect. So first of all, if you look at the differential equation, and when kappa is small, in particular, kappa is smaller than 2 times the vacuum Rabi frequency, then the population to be in the excited state, which is rho 2, 2, undergoes Rabi oscillation. And those Rabi oscillations, this population is damped by e to the minus kappa t.

So what I want to discuss now is the overdamped case. And actually, before I derive it for you, I wanted to ask you some clicker question. I'm afraid I put the books out, but somebody put it away. So nobody took clickers?

So why don't we just pass them around? And I can already formulate the question. I think you know by now that I have a preference to maybe ask you questions about really simple quantum physics. Quantum physics is sadly enough that nobody fully understands it. And we want to improve on that.

So I want to test your intuition by looking at the situation. We have an excited state without photon. We have a ground state. So I want to test your intuition in the following way.

We have Rabi oscillations between two levels. The case in the cavity is, it is Rabi oscillation between state two and state one, but it doesn't matter. You can also assume that it is Rabi oscillation between hyperfine levels. And the Rabi oscillation is driven by an RF field.

And the one thing we introduce now is we introduce damping kappa to this level. But we start out the system with all of the population in this state. And in the overdamped case, when kappa is sufficiently strong, then we know an oscillator will be overdamped and will no longer oscillate.

And in this situation, the probability to be in the initial state will decay. e to the minus kap-- e to the minus-- how should we call it? gamma t. So it decays with the decay rate kappa.

And my question for you is, A, B and C, whether the decay rate, kappa, is proportional to the damping rate, gamma? Proportional to the inverse of gamma? Or independent of gamma?

So we've exacted this system. The system has possibility of Rabi oscillation. I showed you the damped Rabi oscillation and in the limit of small kappa. But now you go to the overdamped regime. And my question is, if the system is overdamped, will this decay rate increase with kappa? Decrease with kappa? Or will it be independent of a damped regime?

All right. OK. There's a clear neutrality. Let me not yet discuss the answer. Let me maybe try to give you another problem. And you should maybe figure out if those two problems are related.

I want to ask you now something simpler. There are not three levels involved, there are two levels involved. But one of the level is broadened by kappa. Call it an unstable state. Because it can decay with the rate, kappa, and we often show something which is broadened.

And now we have a matrix element between the discrete level and the broadened state, which I call omega. It's a matrix element or Rabi frequency. And now you should figure out what is the physical picture which describes the coupling of a discrete initial state to something which is broadened.

It's now the transition rate from the original state. Does the original state decay with a collectivistic rate, which is omega? Or is it omega squared over kappa? Or is it none of the above? So coupling between two levels, very, very simple. But one level is broadened.

OK. Yes, it's Fermi's Golden Rule, what I'm asking you. This is nothing else than Fermi's Golden Rule. We couple from one level into some continuum of states.

If I take one state and smear it out over a width, kappa, the density of states, the density of modes which appears in Fermi's Golden Rule, is 1 over kappa. And Fermi's Golden Rule tells us that the coupling is the matrix element squared times the density of states, which is 1 over kappa. So this is nothing else than Fermi's Golden Rule.

OK. Why don't we come back to the first question now where we have two discrete levels, maybe two hyperfine states, which are coupled by a Rabi frequency? But then one of the hyperfine states decays because there is some leakage, or we-- I don't know-- we interrogate this state with laser beams or such. We make it unstable. We allow this state to decay to another state.

And the question is, if we make the decay of that level stronger and stronger, does the decay of the initial state increase? Decrease? Or is of that? Good.

So what you have learned here is that, actually, the more we damp the system on this side, the longer-lived is the initial state. But it's trivial. It's just Fermi's Golden Rule. When you couple into a level which is broadened, the coupling becomes weaker and weaker because, well, that's what Fermi's Golden Rule tells us.

I want to now derive this situation for you for our atom in the cavity. And I will say that this is one example for the quantum Zeno effect. But let me already explain to you what the quantum Zeno effect is.

Zeno is a Greek philosopher. And the Greek philosophers had some deep thoughts about the nature of time, the nature of motion. And I think, in the Greek philosophy--

sorry. I didn't go-- I mean, I don't have a strong background in philosophy, but I think the idea is that motion is when something moves around. But it takes some time to move.

So if you observe it, it can't move, because every time you observe it, you localize something. That's the idea. If you see an arrow, the moment you observe it, it's localized. And if you localize something to often, it cannot move.

And this is sort of what you can say here. The system wants to evolve to a second state. But kappa can actually be a measurement process. We can just shine a strong laser on this state and figure out has something arrived here. And the more often we look, the stronger our measurement is, the less the system can evolve.

So a measurement of what arrives in this state slows down the dynamics to that state. And this is called the quantum Zeno effect. It also goes by the name of quantum Zeno paradox. But it's just quantum mechanics, it's not a paradox. In the popular world, it is sometimes paraphrased as if you observe the tea kettle, the water never boils. But you see now where it comes from.

OK, so this is the quantum Zeno effect. And we want to now, by looking at the master equation for an atom in a cavity, I want to show you the limit that it really comes out of the master equation. Questions about that?

OK, so this is the quantum Zeno effect. And well, one of the nicest papers on the quantum Zeno effect-- well, I'm biased here-- but it was written by my group, because we used the Bose-Einstein condensate. We had an RF coupling to another hyperfine state.

And then we used a laser beam and observed the population in the final state. And we saw that the effect of the RF drive became weaker and weaker and weaker, the stronger we made the measurement in the final state. And this is the reference.

And our work was the first quantitative comparison. You can now do a measurement by just using a strong laser in, maybe, every millisecond or so. Or you can use a weaker laser beam continuously. So we showed in this work that the quantum Zeno effect is the same, whether you do a pulsed observation, which is a strong measurement, or continuous weak measurement.

OK. But let's now, after this short interlude, go back to the master equation. So our job is now to solve this master equation for the limit of strong damping. And what we will find, actually, is we will find both the quantum Zeno effect and another tidbit. We will also find the Purcell effect, which is enhanced spontaneous emission to a cavity.

Now the way how we solve this master equation in this limiting case is by the method of idiopathic elimination of coherences. And let me explain it to you with that equation. What we have here is the derivative of the coherences. And when you look at the right-hand side, the derivative of the coherences is kappa, a damping terms, times the coherence.

Well, if you have an equation which says a dot equals minus strong damping times a, you have a very rapid exponential damping. The system immediately gets into an equilibrium. And the equilibrium is given by the first term.

So if you have a situation that the population do not rapidly change, then this equation can be written that we have a very rapid damping. And ultimately, the coherences settle to an quasi-equilibrium value in a time 1 over kappa, which is given by the first term. In other words, you set the time derivative of the left-hand side, 0. And then the coherences are expressed in terms of the slowly changing quantity, which are the populations.

So let me repeat. If coherences are rapidly damped, we can neglect the time derivative of the left-hand side. And at any given moment, the coherences will be given by an expression which involves the population. And if the populations slowly change, the coherences will slowly change.

Or in other words, the coherences follow the population. And the lag time is, at most, 1 over kappa. And we are in the limit of strong kappa. This principle of eliminating coherences and we simply get a master equation for population is called the idiopathic elimination of coherences.

It's also called by Eric Hargan in his famous work about synergy by the principle that, if you have rapidly damped modes, they are slaved by the slow modes. In other words, the rapidly damped degrees of freedom follow the slow degrees of freedom. And this is what I just said, that the instantaneous value of the coherences is always given by the population.

So let me just write that down and see where it takes us. So we have the situation that we want to now look at the case of strong damping. This is the overdamped case. And we use now the method of idiopathic elimination of coherences, because they are rapidly damped.

And there is a wonderful discussion about this method in Atom-Photon interaction. It's the Exercise 18 on page 601. So mathematically, what we assume is that the population vary slowly. For instance, they vary at the Rabi frequency.

So therefore, we can say that over the rapid time evolution of the coherences-- rho 1, 2 plus rho 2, 1-- so this is the rapid time evolution of the coherences. But for times which are on the order of 1 over kappa, the population are not changing. The population change over the much longer time scale, which is the period of the Rabi oscillation.

So therefore, for the short time scale, I can sort of pretend that this term is constant. And for obvious reasons, let me now call this constant term, or slowly varying term, it is, at least for short time scales, the equilibrium value.

So the way how we should read it is that the coherences will very, very rapidly damp to this equilibrium value. And this equilibrium value was given by the population, which may slowly change. But that means, now, by eliminating coherences, we obtain a rate equation for the populations only.

So therefore, if you neglect short transients over the short time 1 over kappa, the coherences are always given by this quasi-equilibrium value, which is expressed by the population. And therefore, we can go to our master equation, which was a master equation for the of diagram matrix elements of the density matrix, and we can now replace the coherences by an expression which involves only population. So we get now closed equations for only the population.

I'm sort of emphasizing that because you may have asked yourself, when you have the rate equation a la Einstein, with the a and b coefficient, these are just rate equation for populations. But here, in our class, we've always talked about the density matrix, how important the coherences are. The coherences are important. Without coherences, you would never absorb or emit light. But if the coherences can be expressed by populations, you don't need a differential equation for them.

OK. So with that now, we have our master equation for the population. And by just inserting that, we find this. And we were interested in the decay of rho 1, 1. Or this is also the population of the atom in the excited state.

If we initially start where rho 1, 1 is large, we're interested in the initial decay, then the population in the ground state is small. And we find, indeed, that there is exponential decay of the initial distribution. And the decay rate, gamma z, gamma cavity, is given by this expression. So therefore, what we find is-- but you know it by now-- the effect that the stronger we damp the cavity, the smaller, the weaker, is the decay of the initial state.

And I've already explained to you that one picture to explain that is that we have the excited state, we have the ground state. The ground state gets damped by kappa. And therefore, the time evolution, which is given by the vacuum Rabi oscillation, slows down.

OK. Yes. Any questions? So we have now resolved for gamma cavity. We have a result how the atom decays in the presence of the cavity.

And I want to rewrite this for you in a very nice way. Namely, I want to show you the famous result that, when an atom emits in a cavity, that the atom can decay faster than the spontaneous decay. Because if the cavity has a resonance, the cavity changes the vacuum around the atom. It increases the density of modes. And therefore, there is a stimulation of the atom into the cavity.

So I don't have to do any other math than just rewriting our result of gamma cavity in other units, in other ways. And then we can compare it to the natural decay of an atom without the cavity. So what we want to do now is we want to take the cavity-induced decay of the excited state, and we want to compare with the natural decay rate, gamma.

So I just have to rewrite things in a few units. The cavity-induced decay is the vacuum Rabi oscillation squared over kappa. So let me remind you that the Rabi frequency is always given by an dipole moment times electric field. It's a matrix element.

Well, there's one of those factors of 2. And H bar keeps the units correct. And in the case of the vacuum Rabi oscillation, the electric field is the electric field, so to speak, of one photon in the cavity volume. And there is a factor of 2, so this is now the volume of the cavity. So therefore, the vacuum Rabi frequency is a dipole moment times 2 atomic resonance frequency epsilon 0 h bar v.

And that means now-- and this is a nice result-- that the cavity-induced decay is omega 0 square over kappa. And I just use the above result for omega 0. And for kappa, I introduce the Q factor of the cavity, which is the number of oscillations before it is damped. It's omega over kappa.

And with that, I can rewrite the cavity-induced decay as 2d squared epsilon null H bar v times Q. And if you remember, from part 1 of the course, or standard textbook result, that spontaneous decay is a dipole moment squared times frequency to the power 3, then we obtain the famous result, which is called the Purcell factor, namely that the decay in the cavity, compared to the decay in free space, is proportional to the Q of the cavity. There is a numerical prefector. But then, what enters is the wavelength of the photon cubed over the volume.

So the case of eta larger than 1-- and it's actually interesting that this result is completely independent of atomic properties. So in other words, every atom you put in a cavity, no matter what its dipole moment is, it will decay with a rate which is Q times larger.

There are many ways how to look at it. You can say, if the cavity consists of two mirrors, you create Q mirror images of your atom. And you have sort of an atom which has a Q times stronger dipole moment. And therefore, because the mirror images also radiate, and therefore you get a Q times faster decay.

So this limit makes sense in many, many different pictures. And it's a general result that, if an atom emits in a cavity with finesse Q, you get a Q times enhancement of spontaneous decay. So this is called cavity-enhanced spontaneous emission. It goes back to Purcell in 1946. And you can see the whole field of cavity QED depends on this effect, that you can enhance spontaneous decay.

Or it's the opposite. If an atom has a resonance frequency between two modes of the cavity, the spontaneous decay is inhibited. And this is sort of what is exploited in cavity QED.

Let me just conclude here by saying that our assumptions have to be consistent. In our derivation, we neglected gamma. When I said the two Lindblad operators, we only look at the leakage out of the photon. We neglect spontaneous decay. So of course, this requires that eta is larger 1. So this is the consistent range of validity of our result.

Secondly, we discussed the limit of overdamping, which requires that kappa has to be larger than the vacuum Rabi frequency. And therefore, the above result is valued for large Q. Q has to be sufficiently large, but cannot be larger than this value.

Any questions? Nancy?

AUDIENCE: For kappa, is that the loss in the cavity-- is there any difference between the loss and the cavity because it's a cavity? Or the loss is into other modes of the cavity?

PROFESSOR: Kappa is the loss. You can say kappa is the rate at which a photon in the cavity is lost.

AUDIENCE: Is lost from the cavity totally? Or is lost into another mode inside the cavity? Is there any difference? Is v here just one mode inside the cavity.

PROFESSOR: Well, we have not an element-- OK, if the mirror has imperfections, we usually think of light leaking out of the mirror. If the mirror is absorbing because the coating has a tiny little bit of absorption, you know, from the beam splitter analogy, that a small absorption is equivalent here to a small transmission factor.

Usually, we don't consider that the light transforms from one mode of the cavity to the other. It's either absorbed or leaking out. But if you assume that the mirrors are not super polished and they scatter light a little bit, then you would have a mechanism that the light in the cavity scatters into other modes. And all those losses are summarized in one coupling constant, kappa.

Let me finish the class today and this week by showing to you the recent nature paper by the Innsbrook group. We've talked so much about Lindblad operators. And the Lindblad operators, the dissipation, tells us what is the effect of the environment.

And usually, you would say, well, if you have an atom, and it interacts with the environment, what is the equilibrium state? Well, if you just have the vacuum, the atom goes to the ground state. And the ground state is the attractor, the stable state of the system.

If you drive the system with one laser beam, the attractor state where the system goes into is the steady state solution of the optical Bloch equation. That all sounds a little bit boring. But in this paper, based on this theoretical suggestion of Peter Zola and an experimental realization with trapped ions at Innsbrook, they actually engineered an environment, engineered Lindblad operators in such a way that the system of two ions was relaxing into a Bell state. So you engineer now the environment that therm-- I wanted to say thermal equilibrium, no-- the equilibrium state with the environment, the dissipation, leads to a Bell state.

Well, it's a little bit complicated how it is done, but I just want to just look at a few highlighted key messages. So the idea is, here, engineering of dissipation, creating experimentally dissipative operators. We usually manipulate quantum system with

Hamilton operators. But here, it is manipulated not by the H operator, but by the L operator.

And so the idea here is to have an evolution of the density matrix, a linear mapping that the density matrix evolves in a certain way. I think you'll recognize partial trace. You'll recognize a few equations, a few results which we have discussed.

And what you see here is exactly the Lindblad form. And what they did is, by using a number of laser beams in quantum operation, they designed operators, z, jump operators, Lindblad operators, in such a way that the system was damping out into a Bell state.

So that seems a new frontier for research. It goes by the title that we can do quantum simulations, using trapped ions and cold atoms of Hamiltonians. But we can also quantum simulate very special environments. Maybe environments which it would be hard to find them in nature, but they are quantum-mechanically allowed possible environments. And they may have new features which have not been explored so far. I will add this paper to the website, if you want to read more about it.

Any question? Then a reminder. No class next week. Monday is Patriot's Day. I'm out of town on Wednesday, but you have a homework to solve.

I rather feel you should continue to have a homework assignment every week, because then you have more time at the end of the term for the term paper. I always have 10 homework assignments. So when we spread them out more, there is less time left at the end of the semester for the term paper. So I decided, since we have covered enough material, we have a wonderful homework assignment for next week.