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PROFESSOR: OK. Well, the result of which we obtained on Wednesday for spontaneous emission for the Einstein A coefficient regarded as an accomplishment as a highlight of the course.

We've worked hard to talk about atoms and electromagnetic fields. And ultimately, to deal with spontaneous emission, it was not enough to put a semi-classical light atom interaction, dipole Hamiltonian, Rabi oscillation and such to put that into the picture.

We really needed a quantized version of the electromagnetic field. And this is a result when an atom is excited and interact with all of the empty modes of the vacuum. And be summed up the probability that photon is immediate in any of those modes.

And by doing, kind of, all of the ever reaching with intensity of state, and for all the possibility of actions we obtained. The famous result for the Einstein A coefficient, which is also the natural aligned width of the atomic excited state. Do you have any questions about the derivation or what we did last week?

Then I think I will just continue and interpret the result. So we go to result for an Einstein A coefficient. And well, the question is, how big is it? Well it has a number of constants. And if it is-- let's discuss it now in atomic units.

Well, if we assume the frequency or the energy is on the order of Rydberg-- that's sort of the measure for an electronic excitation in the atom-- we assume the dipole matrix element is one. That means one per radius.

Since we have pretty much set everything one and expressed everything in atomic

units, it means that the speed of light is-- remember? The velocity of the atom in any correspondent was alpha times smaller than the speed of light. But the velocity of the atom is one atomic unit.

So therefore, the speed of light in atomic units is one over alpha. And that means that if you look at the formula, there is the speed of light to the power of 3 in the denominator. And that means that in atomic units the Einstein A coefficient is alpha to the 3, which is 3 times 10 to the minus 7.

So that means that the ratio of this spontaneous emission rate, which is also the inverse lifetime and, therefore, the natural alignments of the excited state. Relative to the transition frequency, so the damping of the harmonic oscillator or the two level system relative to the NFC spacing of the oscillator. It's small. It's actually alpha cube.

So if you take this 3 times 10 to the minus 7 and multiply it with the atomic unit of frequency, which is 2 Rydbergs. We obtain on the order of 10 to 9. And it's a rate of 10 to the 9 per second. And that means that the lifetime of a tubercle atomic level is on the order of 1 nanosecond.

Well, often it's 10 200 nanosecond because many transition frequencies are smaller by quite a factor than the atomic unit of the transition frequency. Remember, the Rydberg frequency would be deep in the UV. But a lot of atoms have transitions in the visible.

I highlighted already when I derived it that the spontaneous emission has this famous omega cube dependents. And that this actually important to understand why lower lying levels-- excited hyperfine levels-- do not radiate. So let me just, kind of, formalize it.

If I would now estimate what is the radiative lifetime for a transition, which is not as I just assumed in the UV or in the visible. Let me estimate what is the radiative lifetime to emit a microwave photon at a few gigahertz? Well, the microwave frequency of the gigahertz 10 to the nine is five orders of magnitude smaller than

the frequency 10 to the 14 of an optical transition.

So therefore, this is 10 to the 15 times longer. And if you have, typically, one or ten nanosecond for an electronic transition. That means that this spontaneous lifetime for a microwave transition is seven months.

If in addition we factor in that hyperfine transitions have an operator which is a Bohr magneton and magnetic type of operator, not an electric dipole and we discussed that the Bohr magneton is actually when we discuss multiple transitions we discuss that the Bohr magneton is alpha times smaller than a typical electric dipole moment.

So therefore, a magnetic dipole transition is alpha times weaker than an electronic dipole transition. And that means now, if you multiply months, which we obtain by the frequency scaling, again, by alpha square for the weakness of the magnetic dipole, we find that atomic hyperfine levels have a lifetime, which is on the order of 1,000 years.

And this is why it's very safe to neglect those transition in the laboratory and assume that all hyperfine states in the ground state manifold pretty much don't decay and are long lived. Questions? OK, so with that we have discussed spontaneous emission.

Let's go through a few clicker questions to discuss the subject and verify your understanding. So the first question is can an E2 transition, which is a quadruple transition, can you drive it by a plane wave? Or does it need a laser beam which has an intensity gradient such as a focus laser beam. Yes or no?

OK. Well the answer is yes. You can just use your laser beam. If a quadrupole transition would require a gradient, it would really require a gradient over the size of the atom. And that would be extremely hard to achieve.

Fortunately, this is not the case because what happens is we actually assumed in the derivation that we had a plane wave into the IKR. And then do the Taylor expansion. And it was these part of the Taylor expansion of a plane wave, which gave rise to the matrix element for the quadrupole transition. So a plane wave laser beam

is sufficient to drive higher multiple transitions.

Next question. Can spontaneous emission be described as a stimulated emission process by the zero point field.

So by the zero point field, we know the electromagnetic wave is a harmonic oscillator. And a harmonic oscillator has a ground state. And in the ground state you have zero point motion. So there is an electric field, even when we have the vacuum state. And the question is, can spontaneous emission be described as simply being stimulated emission but now do to the silver point fluctuations of the electromagnetic field.

OK. The answer is it depends. It depends if you just want to make a qualitative hand waving argument. Then I would say you are correct. You can say that the electromagnetic field of the vacuum stimulates a transition. But when I said described, I meant if you can get it quantitatively correct. And there the answer is actually no because the energy of the electromagnetic field is $n + 1/2 \hbar \omega$.

Whereas, this spontaneous emission of eight is $n + 1$. So you have half a photon verse of extra energy. But this spontaneous emission is sort of like the spontaneous emission is the rate, which would be stimulated by an extra energy of $\hbar \omega$.

So in other words, you would get the answer wrong by a factor of two. I think decoding deeper in the electrodynamics description of spontaneous emission you would identify two terms for spontaneous emission. One is actually the stimulation by the vacuum field. But there is another term called radiation reaction.

So there's, sort of, two terms. Trust me. If not, there are hundreds of pages in [INAUDIBLE], which is books written about it. And in the ground state, the two terms destructively interfere. Therefore, you have no spontaneous emission in the current state, which is reassuring.

But then in the excited state the two terms constructively interfere. And therefore, you get spontaneous emission, which is twice as much as you would get if you just

look at the stimulation by the vacuum field. So the answer is not quantitative but half of it, yes, can be regarded as stimulated emission by the vacuum fluctuations of the electromagnetic field.

OK. We emphasized that spontaneous emission is proportional to omega cube. The question is now what is the dependence in one dimension? If everything the atom can only emit in one dimension, everything is one dimensional, put the atom into a waveguide.

So your choices are omega cube, omega square, or omega-- well, if you press D, none of the above. But I can already tell you it's one of those three. So everything the same. But we are in one dimension.

The world seen by the atom and by the electromagnetic waves is one dimensional. Yes, it's correct. As you remember, out of the omega cube dependence. Omega square came from the density of states. And what is omega square in three dimension becomes omega in two dimensions and constant density of state in one dimension.

So therefore, in one dimension, we are only left with the omega dependence. OK, so there is one factor of omega, which does not come from the density of state. And the next question is where does the other power of omega come from?

As we discussed, it's not the density of states. So we have three choices. One is it comes from the atomic matrix element, it comes from the dipole approximation, or it comes from the quantization of the electromagnetic field.

OK, the majority got it right. It's a field quantization. Sort of remember when you write down the electric dipole Hamiltonian, in the quantized version, there is a perfecter, which is electric field of a single photon. So if you have a single photon, it gives rise to an electric field squared, which is proportionate to $\hbar \omega$. And this is, sort of, the normalization factor.

Two more questions. We talked a lot about the rotating wave approximation. And

we also talked about it for a spinning system driven by magnetic field. If you have a rotating magnetic field, we do not need the rotating wave approximation because if you drive a spin system with a rotating magnetic field, we have only the co-rotating term.

The question I have now for you is whether the same is correct or not for an electronic transition. So therefore, the question is for electronic transitions do we always get the counter rotating term. And if you want to have a simple Hamiltonian, then we do the rotating wave approximation.

So the question is is the rotating wave approximation necessary because we always get the counter rotating term for the electronic transition, then the answer is yes. Or are there examples where the system is exactly described by only one term? The core rotating term.

I will come back to that later in the class. But I thought it's a good question. OK, let me give you the answer. I actually coincide with everybody in the class here because I would tend to say no because there are situations where the counter rotating term can be zero due to angular momentum selection rules.

However, if you have an electronic transition and you have a sigma plus transition to one state, there's always a possibility for sigma minus transition. So you usually get both. But if you apply an infinitely strong magnetic field, then the m equals minus 1 state can be moved out of the picture.

You have only, let's say, the m equals plus 1 state. And then selection holds mean that the counter rotating term is vanishingly small. But it's an artificial situation. So you can all claim credit for your answer.

Finally, the last question is about the Lamb shift. We are now talking about electronic transitions. And the question is Lamb shift-- if it's due to the counter rotating term.

In other words, if you have a situation where the counter rotating term is zero, as we just discussed in the previous example that there may be situations. Somewhat

artificially but you could arrange for it. The set then implies that there is no Lamb shift.

So yes or no. Is the Lamb shift caused by the counter rotating term involved in electronic transitions? OK. OK, well what else is the Lamb shift? It is the AC Stark effect of the counter rotating term.

So is it due to the counter rotating term? Yes, of course. The Lamb shift is the AC Stark effect caused by the vacuum fluctuations. That's what it is.

But we come to that because I want to discuss later today some aspects of the fully quantized Hamiltonian. And we will, again, in the fully quantized picture see the operators, which are responsible for the counter rotating for the counter rotating term. And then I will point to the operator, which causes a Lamb shift. But before I continue, any questions about the questions? Collins.

AUDIENCE: When you derive the amplitude in the electric field due to the single photon--

PROFESSOR: Yep.

AUDIENCE: I always get the factor of two wrong. So you wrote $\hbar\omega$ is $2\epsilon_0$ [INAUDIBLE] squared. Now there's a contribution that comes from the electric field and magnetic field because you have one factor of two. Then there's always that other factor of two. Are you getting that from using one half \hbar because of the vacuum fluctuation.

PROFESSOR: I'm not going back to the formula because I run the risk that it was wrong. But all I want to say is what I really mean is use Jackson. Put in a volume V -- an electromagnetic field-- with $\hbar\omega$ energy.

And the electric field squared of this photon, that's what I mean. And if you find a factor of two mistakes in my E^2 , I can still, you know, get out of theory exit by the rear-entrance door by saying that there is also a difference whether E^2 is E^2_{RMS} or whether E^2 is the amplitude.

You know I mean there are risk factors of two everywhere. But what I mean is really the electric field caused by one photon. And of course, the argument stands. I don't need any factors of two or any subtleties of the electromagnetic field energy. We know that the energy is $n + 1/2$ but emission is $n + 1$. And these shows that the stimulation by the vacuum field cannot quantitatively account for spontaneous emission.

AUDIENCE: So the quantity that you set equal to is $\hbar \omega / 2$, not the fluctuation but the real--

PROFESSOR: OK, if you want to know, let's not compare apples with oranges. You want an electric field. And you can pick whether it's the RMS field or whether it is the maximum amplitude. You can pick what you want.

But now we are comparing what is the e -square for the vacuum-- for single-mode-- vacuum. And what is the e -square for single photon? The two answers differ by a factor of 2. A single photon is twice as strong in e -square as the vacuum fluctuations in the same mode. That's what it means. Yes?

AUDIENCE: I have a question about the quantum emission rate. The explanation that it had-- quantum mechanic derivation that we have, do people not know the formula, how to describe spontaneous emission [INAUDIBLE]?

PROFESSOR: I think so. I have not gone deeply back into the story. But a lot of credit is given to Einstein. And as I mentioned last week that Einstein actually had spontaneous emission in his derivation for the Einstein A and B coefficient in this famous paper. And so he found that there must be spontaneous emission based on a thermodynamic argument.

It's only spontaneous emission, which brings the internal population of an atom into equilibrium. So I think it is correct to say.

AUDIENCE: Can you derive it from that stagnant condition of getting [INAUDIBLE]?

PROFESSOR: That's what Einstein did. And the answer is, by comparison with the Planck law, you

get an expression for the Einstein A and B coefficient. Now of course, you can go the other way around. You can see if you just use classical physics you would actually expect-- now it depends. If you use the Bohr model, you would expect that the electron is radiating and it was a mystery. How can you have an atom in the ground state, which is circling around a nucleolus, and not radiating at all?

On the other hand, in quantum mechanics, we are not assuming that the atom is circulating. And we have an accelerated charge and then we have a time dependent charge distribution. We use the steady state wave function.

So I'm not sure if there is maybe an argument, which would say there should be some spontaneous emission based on a purely classic argument. But this would not be the whole story because a classic argument would then deal with the difficulty. Why is there difference between n equals 1, which does not radiate in n equals 2, which radiates.

So my understanding is that it is only the physics either through the perspective of Einstein by just using equilibration or our microscopic derivation using field quantization, which allows us to understand the phenomenon of a spontaneous emission. Other questions?

OK, then before we talk about some really cute and nice aspects of the fully quantised Hamiltonian, I want to spend a few minutes talking about degeneracy factors. I've already given you my opinion. You should not think in almost all situations about levels, which have a degeneracy.

Just think about states. A state is a state, and it counts as one. And if you have a level which has triple degeneracy, well, it has three states. Just kind of count the states and look at the states.

However, there are formula for which involves degeneracy factors. And just to remind you, when we had the discussion of Einstein's A and B coefficient, the Einstein A coefficient was proportionate to the B coefficient responsible for stimulated emission from the excited to the ground state. But the Einstein B

coefficient for absorption was related to the Einstein B coefficient for stimulated emission by involving these degeneracy factors.

So degeneracies appear and in some formal layer that it makes a lot of sense to use them. So I've always said for a fundamental understanding, you should just assume all degeneracies are one. This is how you can avoid, sort of, some baggage in deriving equations.

And I'm still standing to my statement. I want to show you now a situation where it becomes useful to consider degeneracy factors. So let me give you an example. We can now look at the situation where we have an excited P state and a ground state, which is S.

Or I can look at the opposite situation where we have an S state, which can radiate to a P state. Well by symmetry, the different p states and plus 1 and minus 1 m equals 0 are just connected by spatial rotations.

So therefore, their lifetime of the 3 P states and the rate of spontaneous emission are the same. But if you now assume that you have absorption, you go from the S state to the P state. Then you find that the Einstein B coefficient there are now three possible ways. Not just one polarization or 3 polarization.

And you will find that this is proportional to three times r . However, in this situation, it's a reverse but let me just finish here. So here the natural align rates and the rate of stimulated emission described by the coefficient from the excited state to the ground state is proportionate to R .

Whereas, in the other situation, if you have absorption now, well, each of those levels, there's only one transition, one pass way. Therefore, you will find that the coefficient for absorption is proportionate to R . Whereas, γ and the stimulated emission, which is now B_{SP} , is proportionate to three R because there are three pathways.

So depending what the situation is, you have to be careful. And you would say-- but if it's an S to P transition, it maybe connected by the same matrix element. And

therefore, you would say shouldn't there be align strings, which is independent whether you go from S to P or P to S, which just describes in a natural way what is really the coupling between S and P state?

And yes indeed, there is in the literature some definition of line strings where the lines strings S would be proportionate to the sum of all of the eights between an initial and the final state. And do sum over all.

So therefore, when you use this formula for the line strings, whether you have the situation on the left side or on the right side, you will do always the sum over the 3 possible transitions. So the lines things is the same for both situations. It's just generic for an S to P transition.

So if you use this definition but then you have the situation that spontaneous emission is always given by the line strings but you have to multiply now by the multiplicity of the excited state. If you have a P state, the whole line strings is distributed over three states. And each state has only a spontaneous emission rate, which is a third of what the line strings gives you.

I don't want to beat it to death, because I hate degeneracy factors. But I just thought this example with the P to S and S to P transition tells you why they necessarily have to appear in derivations like Einstein's A and B coefficient. I hope there are no further questions about degeneracies.

But you know, making this comment also allows me to say, well, when I derived the Einstein A coefficient-- what we did last class-- I did not use any degeneracy factors. Well, this is correct. Our derivation assumed that there was-- we assumed that there is only one final state. We did not include degeneracy factors.

We also assumed that we had a dipole matrix element, which was along the z-axis. And so by those definitions, I have implicitly picked a geometry, which can be represented by that we have an exciting piece state in the m equals 0 state. And we have a pie transition with linear polarization to this s state.

And by doing that, I did not have to account for any degeneracies. But in general, if

you derive microscopically an equation for spontaneous emission, you may have to take into account that your excited state has different transitions-- σ plus and σ minus transitions-- to different states. And you have to be careful how you do the sum over all possible finer states.

And this is where degeneracies would eventually matter? Questions? OK, so then lets go from P counting or accounting for the number of states to something, which is hopefully more exciting. We want to talk about the fully quantized Hamiltonian.

So what we are working towards now and it may spill over into the Wednesday class is I want to give you the sort of paradigmatic example of cavity QED where an atom within an excited state is in an empty cavity. And now it can emit a photon into the mortification mode of the cavity.

But these photon can be reabsorbed. So this is a phenomenon of vacuum Rabi oscillations. And so I want to set up the Hamiltonian and then the equation to demonstrate to you the vacuum Rabi oscillations.

And for me, the vacuum Rabi oscillations are the demonstration, that spontaneous emission, has no randomness, no spontaneity, so to speak because you can observe coherent oscillation. A coherent time evolution of the whole system and which is possible only due to spontaneous emission. So let's go there.

So just to make the connection, a few lectures ago, we had a semi classical Hamiltonian. This is when I wanted to show you that the two level electronic system can be mapped onto a spin one half system driven by magnetic field. So this was when we only looked at the stimulated term when we only did perturbation theory.

And in that situation, we had the electronic excitation. And then we had the drive field, which was assumed to be purely classical like a rotating magnetic field which drives spin up spin down transitions magnetically.

And we concluded that, yes, if you use a laser field, it does exactly the same to a two level atom what a magnetic field does to spin up spin down. But now we are one

step further. We've quantized the electromagnetic field. And we have spontaneous emission. And this is something, for reasons I just mentioned, you will never find in spin up spin down because it will take 1,000 years for spontaneous emission to happen.

So now we want to actually go beyond this semi classical picture, which is fully analogous to the precession and rotation of the spin in a magnetic field. And we want to add spontaneous emission. So what we had here is the Rabi frequency was a matrix element-- the dipole matrix element-- times a classic electric field.

And we want to replace that now by the electric field at the position of the atom. But we want to use the fully quantized version of the electric field. And it also becomes useful to look at the sigma x operator, which actually has two matrix elements of [INAUDIBLE], which connects ground excited and excited ground state.

And one of them is going from the excited to the ground state. So this is, sort of, lowering the energy to sigma minus operator. And the other one will be a raising operator. It raises the excitation of the atom. And we will refer to it as sigma plus.

So the electric field is replaced by the operator obtained from the fully quantized picture. Here we have the prefactor, which is the electric field of a single photon or half a photon, whatever. But it's factors of $2r$ square over 2. We have the polarization.

And now if you would take the previous result and would look at it. Well we want to go to the Schrodinger picture. And I mentioned that in the Schrodinger picture the operators are time independent. So we cancelled the e to the $i\omega t$ term.

If you would go to the result we had last week and would simply get rid of the e to the $i\omega t$ term, you would now find operators a and dagger. But they would have factors of i in front of it. That's a equation we had when we derived it. Well I prefer note to use something which looks nicer. Just use a and a dagger.

And you can obtain it by shifting the origin of time. So we're not looking e to the $i\omega t$ or t equals 0. We wait a quarter period into e to the $i\omega t$ just gives us

factors of i , which conveniently cancel the other factors of i . So what I'm doing is just for convenience.

And let me write down that this is in the Schrodinger picture. OK. So we want to absorb all constant by in one constant now, which is the single photon Rabi frequency.

We have the type or matrix element of the atom. There's a dot product with the polarization of the light. And then we have the electric field amplitude of a single photon. $\hbar \omega$ over $2 \epsilon_0 v$.

So this is what appears in the coupling. And we want to write it as $\hbar \omega_1$ over 2. And this ω_1 is the single photon Rabi frequency. And with that, we have now a Hamiltonian, which is really a classic Hamiltonian, written down in the standard form.

It has the excitation energy times the sigma z matrix. It has the single photon Rabi frequency. The single photon Rabi frequency appears. You know, this is the single photon Rabi frequency. But then the operator for the electric field, after getting rid off the i 's, is simply \hbar plus \hbar dagger. \hbar plus \hbar dagger.

So this takes care of the photon field. And the operator which acts on the atoms are the raising and lowering operator sigma plus and sigma minus. And finally, we have the Hamiltonian, which describes the photon field which is $\hbar \omega$ times a dagger a the photon number operator. Any questions? Yes?

AUDIENCE: [INAUDIBLE]?

PROFESSOR: I mean, we are looking at the interaction with an atom, which is at rest at the origin. Therefore, $e^{i\mathbf{k}\cdot\mathbf{r}}$ is 0. We will only consider the spatial dependence $e^{i\mathbf{k}\cdot\mathbf{r}}$ when we allow the atom to move. As long as the atom is stationary for convenience, we put the atom at \mathbf{r} equals 0.

But in 8.422 when we talk about light forces and laser cooling, then it becomes essential to allow the photon to move. And this is actually where the recoil and the

light forces come into play. But as long as we're not interested in light forces, only in the internal dynamics-- calm and excited state-- we can conveniently neglect our spatial dependencies. Other questions?

So this is really a famous Hamiltonian. And you also see how natural the definition of the single-photon Rabi frequency. So we have one half $\hbar \omega$ for the diagonal σ_z matrix. This is the atomic excitation. This is the unperturbed Hamiltonian of the atom. This is the unperturbed Hamiltonian of the photon.

And now the two are coupled. And the coupling is a product of an operator acting on the photon field plus minus one photon. And the other one is an operator acting on the atoms. And it is plus, minus, and atomic excitation. So let me just remind you of that. The sigma plus and sigma minus operator.

The sigma plus is the atomic raising operator, which takes a ground to the excited state. And the sigma minus operator is the atomic lowering operator, which takes the atom from the excited to the ground state. So this is our Hamiltonian. And to hear about space on which this Hamiltonian acts is the product space of the atom.

Direct product of the states of the light. Or in other words, the basis state would be that we use for the atoms. The states which have zero or one quantum of excitation. So we use excited state or ground state.

And for the photon, we can just use the Fock states where the occupation number is n . Questions about that? So it's a very-- just look at it with some enjoyment for a few seconds. I mean, this is a Hamiltonian, which has just a few terms. But what is behind it is, of course, a power of all the definitions. I mean, each symbol has so much meaning. But in the end, by having this formalism of operators quantized electromagnetic field. We can write down-- we can catch many, many aspects or we can, pretty much, fully describe how a two level system interacts a quantized electromagnetic field with that set of equations.

Of course, the fact is not that everything is so simple. The fact is that we have, by understanding the physics, we have skillfully made definitions, which allow us to

write everything down in this compact form.

So often something is simple to write down. But if there's a lot of physics insight, we spend some time in discussing it. And the first thing I want to just point out and discuss is this interaction term.

We have the product of sigma plus and sigma minus with a and a dagger. So what we have here is we have an interaction term. And this interaction part has actually four terms in a very natural way. Well, let me just write them down. It's sigma plus with a. Sigma minus with a dagger. Sigma plus with a dagger. And sigma minus with a.

OK, so let's discuss those. Sigma plus with a dagger. Sigma plus is actually an absorption process. a reduces the photon number by one, and increases the atomic excitation from the ground state to the excited state.

The other term looks naturally, intuitively like emission. The a dagger operator takes us from n to n plus 1. And sigma minus takes us from the excited state to the ground state. So these are the two terms, which we would call intuitive terms because they make sense.

The other terms are somewhat more tricky. Sigma plus and a dagger means we create a photon and we create an excitation. So in other words, it's not that, like the other term, quantum of excitation disappears from the field, appears in the atom, and vice versa.

Sigma plus a dagger means we have an atom excitation takes us from the ground state to the excited state. Plus, we emit a photon at the same time. And sigma minus a dagger means that we go from the excited to the ground state. So we have an atom de-excitation.

And I would say, well, if the atom is de-excited it should emit a photon. But instead, the photon disappears. So we have those processes. The last two are sometimes referred to in the theoretical literature.

They are off shell. Under shell is energy conservation. Off shell means they cannot conserve energy. But nevertheless, these are terms which appear in the operator. But you should be used to if you have often terms in the operator which cannot drive a resonant transition.

When you looked at the DC Stark effect or when we looked at the AC Stark effect for low frequency photons, those low frequency photons cannot excite an atom to the excited state. So they are not causing a transition, but they led to energy shifts in second order perturbation theory. So therefore, those terms this language now cannot drive transitions. They can only drive transitions to virtual states, which would mean they can only appear in second order perturbation theory that you go up to a so-called virtual state but you immediately go down.

And those terms give only rise to shifts. No transitions because you couldn't conserve energy in the transition. But you can do shifts in second order. And one example, which we discussed in the clicker question is that those shifts are actually Lamb shifts. And in other places, especially in the context of microwave fields, they are called Bloch-Steinberg shifts

And let's just look at one specific state. And this is the simplest of all. We have the vacuum no photons. And the atom is in the ground state. If you look at the four possibilities of the interaction term, there is only one non vanishing term. The photon is at the bottom of all possible states. The atom is at the bottom of the possible states.

So when we act with the four terms on it, the only term which contributes is where those is where those are raised because all the others are 0. The only non vanishing term is where we create a virtual atomic excitation and also a virtual excitation of the photon field. And we know that when we have an atom in the ground state in the vacuum that the only manifestation of the electromagnetic field is, of course, not spontaneous emission but the Lamb shift.

So therefore, if you would apply this operator to the bound state of an electron in an atom, the complicated 1s wave function of hydrogen and sum this operator over all

modes of the electromagnetic field. Then you would have done a first principle QED calculation of the lamb shift.

I'm not doing it but you should understand that this operator-- sigma plus a dagger-- is you operator for the Lamb shift. Questions? Yes?

AUDIENCE: [INAUDIBLE]?

PROFESSOR: Oh, no, everything is. If you have a two level system, this Hamiltonian captures everything which appears in nature if you have a two level system interacting with the electromagnetic field. That's it. A radiation reaction is just something we can pull out of here. Stimulated emission we can pull out of here.

The way how vacuum fluctuations create a lamb shift or the way how vacuum fluctuations affect an atom in the excited state, everything is included in here. The question is just can we solve it. And the calculations can get involved. But this is the full QED Hamiltonian for a two level system.

That's a full picture. I mean, that's why I sort of said before be proud of it. You understand the full picture of how two level systems interact with electromagnetic radiation. The only complication is, yes, if you put more levels into it and such and things can get richer and richer. And-- yes, we have also made the dipole approximation, which we're just wondering how critical it is. Well, we use the electric field \mathbf{a} and \mathbf{a}^\dagger , but my gut feeling is it doesn't really matter what we have. Here is the most generic term, which can create and annihilate photons, and we have the \mathbf{a} and \mathbf{a}^\dagger term.

Actually, I don't know what would happen if you don't make the dipole approximation. Well, if you have two levels which are coupled by magnetic dipole, then you have the same situation. It is just your prefactor, the semi photon Rabi frequency, is now α times smaller because of the smaller dipole matrix element.

So I think you can pick, pretty much, any level you want. And this is why I actually discussed matrix elements at the beginning of the unit. For, pretty much, all of the discussion you're going to have, it doesn't really matter what kind of transition you

have as long as the transition creates or annihilates a photon.

And all the physics of the multiplicity of the transition, magnetic, dipole, electric, quadrupole, or whatever just defines what this the semi photon Rabi frequency is. You've put me on the spot, but the only thing which comes to my mind now is if you would formulate QED not in the dipole approximation but through with the p minus a formulation.

Then we have an a -square term. And then we have the possibility that one transition can emit two photons. So that's not included here.

AUDIENCE: So that's higher--

PROFESSOR: This would be something higher order. On the other hand, we can shoulder the canonical transformation that the p minus a formalization with the a -square term is equivalent to dipole approximation. So the question whether you have a transition which emits two photons simultaneously or two photons sequentially eventually by going through an immediate state, this is not a fundamental distinction.

You can have one description of your quantum system via two photons automated in one transition. You have another description of your quantum system where photons cannot-- only one photon can be emitted. And then you have to lend an intermediate state.

And you would say, well, either two photons at once or one photon at a time. This is two different kinds of physics. But we can show that the two pictures are connected with economical transformation. So therefore, you have two descriptions here.

But anyway, I'm going a little bit beyond my knowledge. I'm just telling you bits and pieces I know. But this Hamiltonian is either generally exact. I just don't know how to prove it. But it really captures in all of the QED aspects of the system we want to get into.

So OK. So in many situations we may decide that the off shell terms of the interaction just create level shifts, Lamb shifts, Bloch-Siegert shifts. And we may

simply absorb those Lamb shifts in our atomic energy levels, ω_e and ω_g .

So therefore, for the dynamic of the system, if you include all of those Lamb shifts in the atomic description, you do not need those off shell counter intuitive terms.

These are actually also the counter-rotating terms in the semi classical approximation. We only keep the intuitive terms. And that's called, again, the rotating wave approximation.

Just to remind you, we do not have rotating waves here. Everything is operators.

But the same kind of physics-- co- and counter-rotating-- appears here that we have four terms. Two are the fully quantized version of the co-rotating terms. And the other two-- the off shell terms-- are the quantized version of the counter rotating term.

So therefore, if you neglect those two off shell terms, we have now the fully quantized Hamiltonian in the rotating wave approximation. So let me just write it down because it's also a beautiful line. We have the electronic system. We have the interaction Hamiltonian, which has now owned the two terms.

When we raise the atomic excitation, we lower the photon excitation and vice versa. And we have the Hamiltonian for the photon field $a^\dagger a$. And this is apart from those Lamb shift terms. The full QED description of the system.

And if we only consider one mode-- here, of course, in general, the general Hamiltonian has to be sent over a modes. And then you'll get spontaneous emission and everything we want. But if you have a situation where you only look at one single mode, then you have what is called the famous chains Cummings model

And very important result of this James Cummings model are the vacuum Rabi oscillations, which I want to discuss now. OK. So let me just-- it's called James Cummings Model. So let me describe to you why it is a model. Well it assumes a two level system, which we find a lot of candidates among the atoms we want.

Sure our atoms have hyperfine states. But we can always select a situation where, essentially, we only couple two states. We can prepare initial state by optical

pumping, and then use circularly polarized light on a cycling transition. And this is how we prepare in the laboratory a two level system.

So that's one assumption of this model with a two level system. But the second assumption is that the atom only interacts with a similar mode. And that requires a little bit of engineering because it means we need a cavity.

So let me just set up the system. So our laboratory is a big box of volume v . And this is where we maybe quantize electromagnetic field to calculate spontaneous emission. And our atom here may actually decay with the rate γ , which is given by the Einstein A coefficient.

And in order to describe this spontaneous emission, we quantize electromagnetic field in the large volume v . But now we have a cavity with two mirrors. And those two mirrors define one mode of the electromagnetic field, which will be in resonance on your resonance with the atom.

Well there will be some losses out of the cavity, which eventually couple the electromagnetic mode inside the cavity to the other allowed modes in the speaker volume v . And this is described by a cavity damping constant κ . What is also important is when we use cavity to single out one mode of the electromagnetic field, the cavity volume is v' .

And we often make it very small by putting the atoms in the cavity where the mirror spacing is extremely small. OK. We know, and I'm not writing it down again, what the Einstein A coefficient is.

The Rabi frequency-- the single photon Rabi frequency-- which couples the atom to the one mode of the cavity has this important parameter, which was or is the electric field of one photon in the cavity.

And importantly, it involves the electric field of the photon in the cavity value, which is B' . So now in addition to using, you know-- now you see what our experimental handle is. If you make this volume very small, then we can enter this strong coupling regime where the single photon Rabi frequency for this one mode

selected by the cavity becomes much larger than the spontaneous emission into all the many other modes.

So the interaction with one mode due to the cavity and the smallness of the volume is, sort of, outperforming all these many, many modes of the surroundings. And that would mean that an atom in an excited state is more likely to emit into the mode between the two cavity mirrors than to any other modes to the side.

Secondly, of course, when the photon has been emitted into the cavity, the photon can still couple to the other modes by cavity losses κ . And now we assume that we have such high reflectivity mirrors that κ is smaller than the single photon Rabi frequency.

And this is called the strong coupling regime of cavity QED. So then we can at least observe for a limited time the interplay between a single mode of the cavity and a two level system. And this is a James Cummings model. The James Cummings model.

So in that situation, the Hamiltonian, the fully quantized Hamiltonian, and the QED Hamiltonian couples only pairs of states which we label those states the manifold n . So we have an excited state with n photons.

And it is coupled to the ground state with one more photon. Our Hamiltonian has two coupling terms. Remember the other tool where you clicked it in the rotating wave approximation and we can go from left to right with σ_- and a^\dagger . And we can go from right to left with the operator σ_+ and the annihilation of the [INAUDIBLE] a .

So as long as we have a detuning δ , which is relatively small. As long as detuning is small, the rotating wave approximation is excellent. So let me just conclude by writing down the Hamiltonian for the situation I just discussed. And then we'll discuss the Hamiltonian on Wednesday.

So if this is energy, we have two levels. The excited state with n photons, the

ground state with $n + 1$ photons. If the photons are on resonance, the two levels are degenerate. But if you have a detuning δ , the two levels are split by δ .

And what we are doing right now is for the [INAUDIBLE] the Hamiltonian, we shift the origin so the zero of the energy is just halfway between those two states. That's natural. So this avoids just off sets in our equations. So our Hamiltonian has now the splitting of $\pm \delta / 2$.

The coupling has the perfecter, which is the single photon Rabi frequency. And then the a and a^\dagger terms depends on $\sqrt{n + 1}$. So what I wrote down now is the Hamiltonian rotating wave approximation, which interacts, which describes only one pair of states.

But we have sort of a cause in our Hilbert space. One pair of states for each label n . But each of them is, sort of, described by the decoupled Hamiltonian. So that's what I wanted to present you today. And I will show you on Wednesday how this Hamiltonian needs to Rabi oscillations not induced by an external field but induced by the vacuum. Any questions?