8.421 Spring 2012 Assignment #3
Professor W. Ketterle
Due Friday, March 9, 2012

1. **Energy shifts in hydrogen due to the size of the proton** (6 pts.)

   (a) Derive the potential produced by a uniformly charged sphere

   $$\rho = \begin{cases} \rho_0 & (r < a) \\ 0 & (r > a) \end{cases}$$

   (b) Calculate the level shift for the $1S$ state of a hydrogen due to the finite proton radius using first order perturbation theory. Assume that the proton has a uniform charge distribution over $r_p = 0.9$ fm.

   (c) What frequency accuracy is needed for $1S - 2S$ spectroscopy to measure r_p with 0.010 fm accuracy? Give your answer in both absolute and relative terms. (Note: you should also consider the level shift for the $2S$ state due to finite size of the proton.)

   Recent measurement of the hydrogen $1S - 2S$ transition frequency (Th. Udem *et.al.*, PRL **79**, 2646 (1997)) had this accuracy and indicated that the rms proton charge radius should be 0.890(14)fm instead of the accepted value 0.862(12)fm. However, in the end it turned out that the discrepancy was due to QED calculations which subsequently revealed a higher-order term which was larger than expected.

   You may find some of the following information useful.

   $$[-\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{r}]\psi = E\psi$$

   is satisfied by

   $$\psi_{1s} = \frac{1}{\sqrt{\pi}}(\frac{Z}{a_0})^{\frac{3}{2}} e^{-\frac{Z}{a_0}r}, \qquad E_{1s} = -Z^2\frac{e^2}{2a_0}$$

   $$a_0 = \frac{\hbar^2}{m_e e^2} = 5.3 \times 10^{-11} \text{ m}, \qquad \frac{e^2}{2a_0} = 13.606 \text{ eV}$$

2. **Atoms in magnetic fields: the Breit-Rabi formula** (10 pts.)

The Hamiltonian for an atom in a magnetic field along $\hat{z}$ may be written

$$H = ah\frac{\vec{I}\cdot\vec{J}}{\hbar^2} + \left(g_J\mu_0 m_J - g_I\mu_0 m_I\right)B_z$$

(a) Restrict attention to the case $J = 1/2$, but arbitrary I. Show that the energies of states are given by the Breit-Rabi formula

$$E_m^{\pm} = -\frac{ah}{4} - m g_I \mu_0 B_z \pm \frac{ahF^+}{2}\sqrt{1 + \frac{2mx}{F^+} + x^2}$$

The parameter x is given by $x = (g_I + g_J)\,\mu_0 B_z/ahF^+$, where $F^+ = I + 1/2$. m is the z component of the total angular momentum.

(b) For the case of $I = 3/2$, make a clear sketch of energies vs. x. Take advantage of the non-crossing rule (levels of the same m do not cross). Be sure to extend your figure to very high field ($1/x \ll g_I/g_J$). Label the lines with quantum numbers at low and high fields and indicate m. You may use the values of ^{87}Rb, which has $I = 3/2$, $g_I = 0.0009954$, and $g_J = 2.002331$.

(c) There are some values of magnetic field x where the resonance frequency ΔE for the (magnetic) dipole transition (selection rules $\Delta m = 0, \pm 1$) is first-order field independent (i.e., the leading order term in ΔE is of $O((\Delta x)^2)$). Show those values for magnetic field and corresponding transitions on your figure from part b).

(d) Which of the transitions you just found connect states that can be confined in a magnetic trap, i.e. a field configuration with a local minimum in the magnitude of the magnetic field? Note that Maxwell's equations forbid a static local maximum of the magnetic field in free space.

(e) The first-order insensitive transition(s) between trappable states you have just found occur(s) for states with a relatively weak dipole moment (dependence of state energy on magnetic field) and at large offset fields. This makes for weak magnetic traps: it is hard to generate large field gradients (needed for tight trapping, particularly of weak dipoles) at a large offset field. The transition $|F = 1, m_F = -1\rangle \leftrightarrow |F = 2, m_F = 1\rangle$ in ^{87}Rb is not an allowed magnetic dipole transition ($\Delta m = 2$), but it can be driven as a two-photon transition using one of the $m = 0$ sublevels as a virtual intermediate state. Like the examples you have looked at in the preceding questions, this transition is first-order insensitive to field fluctuations at some properly-chosen offset field. However, in this case the correct offset field is small, and the two states have large ($\mu_0/2$) magnetic dipole moments. This makes it valuable for precision spectroscopy of trapped atomic samples and for efforts to make compact neutral-atom clocks. For examples, see G.K. Campbell *et al.*, Science *313*:649–652 (2006), D.S. Hall *et al.*, PRL

81:1543 (1998), D.M. Harber *et al.*, PRA **66**:053616 (2002). Calculate the magnetic field for which the transition is first-order field-insensitive. In ^{87}Rb, $I = 3/2$, $2a = 6.835$ GHz, $g_I = 0.0009954$, and $g_J = 2.002331$ in the ground ($5^2S_{1/2}$) state.

(f) ^{87}Rb atoms magnetically trapped in the $|F = 1, m = -1\rangle$ state at 1 μK temperature are distributed over a magnetic field range of about 30 mG. Calculate the inhomogeneous width of the $|F = 1, m = -1\rangle \leftrightarrow |F = 2, m = 1\rangle$ transition at zero magnetic field and at the magnetic field found in the previous question.

3. **Atomic g factors** (4 pts.) Find g factors for the following states of Na ($I = 3/2$):

$^2P_{1/2}$	$F = 1, 2$
$^2P_{3/2}$	$F = 0, 1, 2, 3$
$^2S_{1/2}$	$F = 1, 2$

Can you find the g factors for the states of maximum angular momentum (so-called stretched states $^2P_{3/2}, F = 3$ and $^2S_{1/2}, F = 2$) without resorting to the formula for the g factor derived in class?

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