VI.H Critical behavior of the two-dimensional Ising model

To understand the singularity in the free energy of the two dimensional Ising model in eq.(VI.73), we start with the simpler expression obtained by the unrestricted sum over phantom loops in sec.VI.F. Specializing eq.(VI.53) to $d = 2$,

$$f_G = \ln \left(2 \cosh^2 K \right) - \frac{1}{2} \int \frac{dq_x dq_y}{(2\pi)^2} \ln \left[1 - 2t \left(\cos q_x + \cos q_y \right) \right]. \tag{VI.75}$$

Apart from the argument of the logarithm, this expression is similar to the exact result. Is it possible that such similar functional forms lead to distinct singular behaviors? The singularity results from the vanishing of the argument of the logarithm at $t_c = 1/4$. In the vicinity of this point we make an expansion as in eq.(VI.50),

$$A_G(t, \mathbf{q}) = (1 - 4t) + tq^2 + \mathcal{O}(q^4) \approx t_c \left(q^2 + 4\frac{\delta t}{t_c} \right), \tag{VI.76}$$

where $\delta t = t_c - t$. The singular part of eq.(VI.75) can be obtained by focusing of the behavior of the integrand as $\mathbf{q} \to \mathbf{0}$, and replacing the square Brillouin zone for the range of the integral with a circle of radius $\Lambda \approx 2\pi$,

$$
\begin{aligned}
f_{\text{sing.}} &= -\frac{1}{2} \int_0^{\Lambda} \frac{2\pi q dq}{4\pi^2} \ln \left(q^2 + 4\frac{\delta t}{t_c} \right) \\
&= -\frac{1}{8\pi} \left[\left(q^2 + 4\frac{\delta t}{t_c} \right) \ln \left(\frac{q^2 + 4\delta t/t_c}{e} \right) \right]_0^{\Lambda}.
\end{aligned}
\tag{VI.77}
$$

Only the expression evaluated at $q = 0$ is singular, and

$$f_{\text{sing.}} = \frac{1}{2\pi} \left(\frac{\delta t}{t_c} \right) \ln \left(\frac{\delta t}{t_c} \right). \tag{VI.78}$$

The resulting heat capacity, $C_G \propto \partial^2 f_G / \partial^2 t$, diverges as $1/\delta t$. Since eq.(VI.75) is not valid for $t > t_c$, we cannot obtain the behavior of heat capacity on the low temperature side.

For the exact result of eq.(VI.73), the argument of the logarithm is

$$A^*(t, \mathbf{q}) = \left(1 + t^2 \right)^2 - 2t \left(1 - t^2 \right) \left(\cos q_x + \cos q_y \right). \tag{VI.79}$$

The minimum value of this expression, for $\mathbf{q} = \mathbf{0}$, is

$$A^*(t, \mathbf{0}) = \left(1 + t_c^2 \right)^2 - 4t_c \left(1 - t_c^2 \right) = \left(1 - t_c^2 \right)^2 + 4t_c^2 - 4t_c \left(1 - t_c^2 \right) = \left(1 - t_c^2 - 2t_c \right)^2. \tag{VI.80}$$

Since this expression (and hence the argument of the logarithm) is always non–negative, the integral exists for all values of t. As required, unlike eq.(VI.75), the exact result is valid at *all* temperatures. There is a singularity when the argument vanishes for

$$t_c^2 + 2t_c - 1 = 0, \qquad \Longrightarrow \qquad t_c = -1 \pm \sqrt{2}. \tag{VI.81}$$

The positive solution describes a ferromagnet, and leads to a value of $K_c = \ln\left(\sqrt{2}+1\right)/2$, in agreement with the duality arguments of sec.VI.D. Setting $\delta t = t - t_c$, and expanding eq.(VI.79) in the vicinity of $\mathbf{q} \to 0$ gives

$$
\begin{aligned}
A^*(t, \mathbf{q}) &\approx [(-2t_c - 2)\delta t]^2 + t_c(1 - t_c^2)q^2 + \cdots \\
&\approx 2t_c^2 \left[q^2 + 4 \left(\frac{\delta t}{t_c}\right)^2 \right].
\end{aligned}
\tag{VI.82}
$$

The important difference from eq.(VI.76) is that $(\delta t/t_c)$ appears at quadratic order. Following the steps in eqs.(VI.77) and (VI.78), the singular part of the free energy is

$$
\begin{aligned}
\left.\frac{\ln Z}{N}\right|_{\text{sing.}} &= \frac{1}{2} \int_0^\Lambda \frac{2\pi q dq}{4\pi^2} \ln\left[q^2 + 4\left(\frac{\delta t}{t_c}\right)^2 \right] \\
&= \frac{1}{8\pi} \left[\left(q^2 + 4\left(\frac{\delta t}{t_c}\right)^2 \right) \ln\left(\frac{q^2 + 4(\delta t/t_c)^2}{e} \right) \right]_0^\Lambda \\
&= -\frac{1}{\pi} \left(\frac{\delta t}{t_c}\right)^2 \ln\left|\frac{\delta t}{t_c}\right| + \text{analytic terms.}
\end{aligned}
\tag{VI.83}
$$

The heat capacity is obtained by taking two derivatives and diverges as $C(\delta t)_{\text{sing.}} = A_\pm \ln|\delta t|$. The logarithmic singularity corresponds to the limit $\alpha = 0$; the peak is symmetric, characterized by the amplitude ratio $A_+/A_- = 1$.

The exact partition function of the Ising model on the square lattice was originally calculated by Lars Onsager in 1944 (Phys. Rev. **65**, page 117). Onsager used a $2^L \times 2^L$ transfer matrix to study a lattice of width L. He then identified various symmetries of this matrix which allowed him to diagonalize it and obtain the largest eigenvalue as a function of L. For any finite L, this eigenvalue is non-degenerate as required by Frobenius's theorem. In the limit $L \to \infty$, the top two eigenvalues become degenerate at K_c. The result in this limit naturally coincides with eq.(VI.74). Onsager's paper is quite long and complicated, and regarded as a *tour de force* of mathematical physics. A somewhat streamlined version of this solution was developed by B. Kaufman (Phys. Rev. **76**, 1232 (1949)) and is

more or less reproduced in chapter 15 of Huang. This exact solution, for the first time, demonstrated that critical behavior at a phase transition can in fact be quite different from predictions of Landau (mean–field) theory. It took almost three decades to reconcile the two results by the renormalization group.

The graphical method presented in this section was originally developed by Kac and Ward (Phys. Rev. **88**, 1332 (1952)). The main ingredient of the derivation is the result that the correct accounting of the paths can be achieved by including a factor of (-1) for each intersection. (This conjecture by Feynman is proved in S. Sherman, J. Math. Phys. **1**, 202 (1960).) The change of sign is reminiscent of the exchange factor between fermions. Indeed, Schultz, Mattis, and Lieb (Rev. Mod. Phys. **36**, 856 (1964)) describe how the Onsager transfer matrix can be regarded as a Hamiltonian for the evolution of non-interacting fermions in one dimension. The Pauli exclusion principle prevents intersections of the world-lines of these fermions in two dimensional space-time.

In addition to the partition function, the correlation functions $\langle \sigma_{\mathbf{i}} \sigma_{\mathbf{j}} \rangle$ can also be calculated by summing over paths (see Itzykson and Drouffe, *Statistical field theory: 1*). Since the combination $q^2 + 4(\delta t/t_c)^2$ in eq.(VI.82) describes the behavior of these random walks, we expect a correlation length $\xi \sim |t_c/\delta t|$, i.e. diverging with an exponent $\nu = 1$ on both sides of the phase transition with an amplitude ratio of unity. The exponents α and ν are related by the hyperscaling identity $\alpha = 2 - 2\nu$. The critical correlations at t_c are more subtle and decay as $\langle \sigma_{\mathbf{i}} \sigma_j \rangle_c \sim 1/|\mathbf{i} - \mathbf{j}|^\eta$, with $\eta = 1/4$. Integrating the correlation functions yields the susceptibility, which diverges as $\chi_\pm \simeq C_\pm |\delta t|^{-\gamma}$, with $\gamma = 7/4$, and $C_+/C_- = 1$. A rather simple expression for the zero field magnetization,

$$m = \left(1 - \sinh^{-4}(2K)\right)^{1/8}, \qquad \text{(VI.84)}$$

was presented without proof by Onsager. A rather difficult derivation of this result was finally given by C.N. Yang (Phys. Rev. **85**, 808 (1952)). The exponent $\beta = 1/8$ satisfies the required exponent identities.

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