Review Problems & Solutions

The test is 'closed book,' but if you wish you may bring a one-sided sheet of formulas. The intent of this sheet is as a reminder of important formulas and definitions, and not as a compact transcription of the answers provided here. The test will be composed entirely from a subset of the following problems **as well as those in problem sets 5 and 6**. Thus if you are familiar and comfortable with these problems, there will be no surprises!

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1. *Continuous spins:* In the standard $\mathcal{O}(n)$ model, n component unit vectors are placed on the sites of a lattice. The nearest neighbor spins are then connected by a bond $J\vec{s}_i \cdot \vec{s}_j$. In fact, if we are only interested in universal properties, any generalized interaction $f(\vec{s}_i \cdot \vec{s}_j)$ leads to the same critical behavior. By analogy with the Ising model, a suitable choice is

$$\exp\left[f(\vec{s}_i \cdot \vec{s}_j)\right] = 1 + (nt)\vec{s}_i \cdot \vec{s}_j,$$

resulting in the so called *loop model*.

(a) Construct a high temperature expansion of the loop model (for the partition function Z) in the parameter t, on a two-dimensional *hexagonal* (honeycomb) lattice.

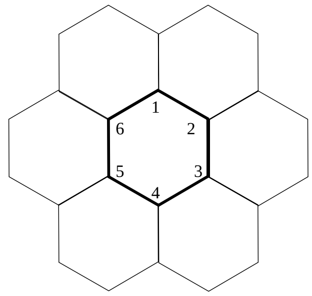
• The partition function for the loop model has the form

$$Z = \int \{\mathcal{D}\mathbf{s}_i\} \prod_{\langle ij \rangle} \left[1 + (nt)\mathbf{s}_i \cdot \mathbf{s}_j\right],$$

that we can expand in powers of the parameter t. If the total number of nearest neighbor bonds on the lattice is N_B, the above product generates 2^{N_B} possible terms. Each term may be represented by a graph on the lattice, in which a bond joining spins i and j is included if the factor $\mathbf{s}_i \cdot \mathbf{s}_j$ appears in the term considered. Moreover, each included bond carries a factor of nt. As in the Ising model, the integral over the variables $\{\mathbf{s}_i\}$ leaves only graphs with an even number of bonds emanating from each site, because

$$\int d\mathbf{s}\ s_\alpha = \int d\mathbf{s}\ s_\alpha s_\beta s_\gamma = \cdots = 0.$$

In a honeycomb lattice, as plotted below, there are only 1, 2, or 3 bonds emerging from each site. Thus the only contributing graphs are those with two bonds at each site, which, as any bond can only appear once, are closed *self-avoiding loops*.



While the honeycomb lattice has the advantage of not allowing intersections of loops at a site, the universal results are equally applicable to other lattices.

We shall rescale all integrals over spin by the n-dimensional solid angle, such that $\int d\mathbf{s} = 1$. Since $s_\alpha s_\alpha = 1$, it immediately follows that

$$\int d\mathbf{s}\ s_\alpha s_\beta = \frac{\delta_{\alpha\beta}}{n},$$

resulting in

$$\int d\mathbf{s}'\ (s_\alpha s'_\alpha)(s'_\beta s''_\beta) = \frac{1}{n} s_\alpha s''_\alpha.$$

A sequence of such integrals forces the components of the spins around any loop to be the same, and there is a factor n when integrating over the last spin in the loop, for instance

$$\int \{\mathcal{D}\mathbf{s}_i\}\ (s_{1\alpha}s_{2\alpha})(s_{2\beta}s_{3\beta})(s_{3\gamma}s_{4\gamma})(s_{4\delta}s_{5\delta})(s_{5\eta}s_{6\eta})(s_{6\nu}s_{1\nu}) = \frac{\delta_{\alpha\beta}\delta_{\beta\gamma}\delta_{\gamma\delta}\delta_{\delta\eta}\delta_{\eta\nu}\delta_{\alpha\nu}}{n^6} = \frac{n}{n^6}.$$

Since each bond carrier a factor of nt, each loop finally contributes a factor $n \times t^\ell$, where ℓ is the number of bonds in the loop. The partition function may then be written as

$$Z = \sum_{\text{self}-\text{avoiding loops}} n^{N_\ell} t^{N_b},$$

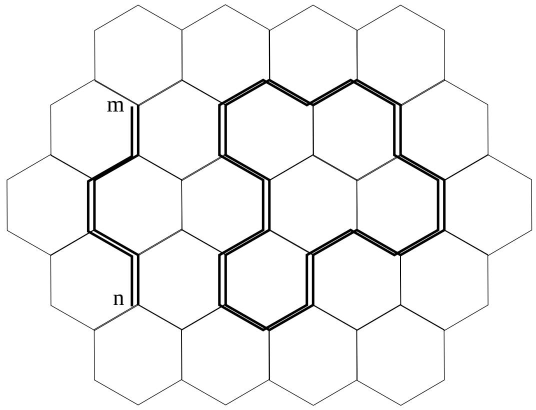
where the sum runs over distinct disconnected or self-avoiding loops collections with a bond fugacity t, and N_ℓ, N_b are the number of loops, and the number of bonds in the graph, respectively. Note that, as we are only interested in the critical behavior of the model, any global analytic prefactor is unimportant.

(b) Show that the limit $n \to 0$ describes the configurations of a single self-avoiding polymer on the lattice.

• While $Z = 1$, at exactly $n = 0$, one may obtain non-trivial information by considering the limit $n \to 0$. The leading term ($\mathcal{O}(n^1)$) when $n \to 0$ picks out just those configurations with a single self-avoiding loop, i.e. $N_\ell = 1$.

The correlation function can also be calculated graphically from

$$G_{\alpha\beta}(n-m) = \langle s_{n\alpha} s_{m\beta} \rangle = \frac{1}{Z} \int \{\mathcal{D}\mathbf{s}_i\} s_{n\alpha} s_{m\beta} \prod_{\langle ij \rangle} \left[1 + (nt)\mathbf{s}_i \cdot \mathbf{s}_j\right].$$



After disregarding any global prefactor, and taking the limit $n \to 0$, the only surviving graph consists of a single line going from n to m, and the index of all the spins along the line is fixed to be the same. All other possible graphs disappear in the limit $n \to 0$. Therefore, we are left with a sum over self-avoiding walks that go from n to m, each carrying a factor t^ℓ, where ℓ indicates the length of the walk. If we denote by $W_\ell(R)$ the number of self-avoiding walks of length ℓ whose end-to-end distance is R, we can write that

$$\sum_\ell W_\ell(R) t^\ell = \lim_{n \to 0} G(R).$$

As in the case of phantom random walks, we expect that for small t, small paths dominate the behavior of the correlation function. As t increases, larger paths dominate the sum, and, ultimately, we will find a singularity at a particular t_c, at which arbitrarily long paths become possible.

Although we presented the mapping of self-avoiding walks to the $n \to 0$ limit of the $\mathcal{O}(n)$ model for a honeycomb lattice, the critical behavior should be universal, and therefore independent of this lattice choice. What is more, various scaling properties of self-avoiding walks can be deduced from the $\mathcal{O}(n)$ model with $n \to 0$. Let us, for instance, characterize the mean square end-to-end distance of a self-avoiding walk, defined as

$$\langle R^2 \rangle = \frac{1}{W_\ell} \sum_R R^2 W_\ell(R),$$

where $W_\ell = \sum_R W_\ell(R)$ is the total number of self-avoiding walks of length ℓ.

The singular part of the correlation function decays with separation R as $G \propto |R|^{-(d-2+\eta)}$, up to the correlation length ξ, which diverges as $\xi \propto (t_c - t)^{-\nu}$. Hence,

$$\sum_R R^2 G(R) \propto \xi^{d+2-(d-2+\eta)} = (t_c - t)^{-\nu(4-\eta)} = (t_c - t)^{-\gamma - 2\nu}.$$

We noted above that $G(t,R)$ is the generating function of $W_\ell(R)$, in the sense that $\sum_\ell W_\ell(R)t^\ell = G(t,R)$. Similarly $\sum_\ell W_\ell t^\ell$ is the generating function of W_ℓ, and is related to the susceptibility χ, by

$$\sum_\ell W_\ell t^\ell = \sum_R G(R) = \chi \propto (t_c - t)^{-\gamma}.$$

To obtain the singular behavior of W_ℓ from its generating function, we perform a Taylor expansion of $(t_c - t)^{-\gamma}$, as

$$\sum_\ell W_\ell t^l = t_c^{-\gamma}\left(1 - \frac{t}{t_c}\right)^{-\gamma} = t_c^{-\gamma}\sum_\ell \frac{\Gamma(1-\gamma)}{\Gamma(1+\ell)\Gamma(1-\gamma-\ell)}\left(\frac{t}{t_c}\right)^\ell,$$

which results in

$$W_\ell = \frac{\Gamma(1-\gamma)}{\Gamma(1+\ell)\Gamma(1-\gamma-\ell)}\, t_c^{-\ell-\gamma}.$$

After using that $\Gamma(p)\Gamma(1-p) = \pi/\sin p\pi$, considering $\ell \to \infty$, and the asymptotic expression of the gamma function, we obtain

$$W_\ell \propto \frac{\Gamma(\gamma+\ell)}{\Gamma(1+\ell)}\, t_c^{-\ell} \propto \ell^{\gamma-1} t_c^{-\ell},$$

and, similarly one can estimate $\sum_R R^2 W_\ell(R)$ from $\sum_R R^2 G(R)$, yielding

$$\left\langle R^2 \right\rangle \propto \frac{\ell^{2\nu+\gamma-1} t_c^{-\ell}}{\ell^{\gamma-1} t_c^{-\ell}} = \ell^{2\nu}.$$

Setting $n = 0$ in the results of the ϵ-expansion for the $\mathcal{O}(n)$ model, for instance, gives the exponent $\nu = 1/2 + \epsilon/16 + \mathcal{O}(\epsilon^2)$, characterizing the mean square end-to-end distance of a self-avoiding polymer as a function of its length ℓ, rather than $\nu_0 = 1/2$ which describes the scaling of phantom random walks. Because of self-avoidance, the (polymeric) walk is swollen, giving a larger exponent ν. The results of the first order expansion for $\epsilon = 1$, 2, and 3, in $d = 3$, 2, and 1 are 0.56, 0.625, and 0.69, to be compared to 0.59, 3/4 (exact), and 1 (exact).

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2. *Potts model I:* Consider Potts spins $s_i = (1, 2, \cdots, q)$, interacting via the Hamiltonian $-\beta\mathcal{H} = K\sum_{<ij>} \delta_{s_i,s_j}$.

(a) To treat this problem graphically at high temperatures, the Boltzmann weight for each bond is written as

$$\exp\left(K\delta_{s_i,s_j}\right) = C(K)\left[1 + T(K)g(s_i,s_j)\right],$$

with $g(s,s') = q\delta_{s,s'} - 1$. Find $C(K)$ and $T(K)$.

• To determine the two unknowns $C(K)$ and $T(K)$, we can use the expressions

$$\begin{cases} e^K = C\left[1+T(q-1)\right] & \text{if} \quad s_i = s_j \\ 1 = C\left[1-T\right] & \text{if} \quad s_i \neq s_j \end{cases},$$

from which we obtain

$$T(K) = \frac{e^K - 1}{e^K + q - 1}, \qquad \text{and} \qquad C(K) = \frac{e^K + q - 1}{q}.$$

(b) Show that

$$\sum_{s=1}^{q} g(s,s') = 0 \ , \ \sum_{s=1}^{q} g(s_1,s)g(s,s_2) = qg(s_1,s_2) \ , \ \text{and} \ \sum_{s,s'}^{q} g(s,s')g(s',s) = q^2(q-1).$$

• Moreover, it is easy to check that

$$\sum_{s=1}^{q} g(s,s') = q - 1 - (q-1) = 0,$$
$$\sum_{s=1}^{q} g(s_1,s)g(s,s_2) = \sum_{s=1}^{q} \left[q^2\delta_{s_1 s}\delta_{s_2 s} - q(\delta_{s_1 s} + \delta_{s_2 s}) + 1\right] = q\left(q\delta_{s_1 s_2} - 1\right) = qg(s_1,s_2),$$
$$\sum_{s,s'=1}^{q} g(s,s')g(s,s') = \sum_{s,s'=1}^{q} \left[q^2\delta_{ss'}\delta_{ss'} - 2q\delta_{ss'} + 1\right] = q^3 - 2q^2 + q^2 = q^2(q-1).$$

(c) Use the above results to calculate the free energy, and the correlation function $\langle g(s_m, s_n)\rangle$ for a one–dimensional chain.

• The factor $T(K)$ will be our high temperature expansion parameter. Each bond contributes a factor $Tg(s_i, s_j)$ and, since $\sum_s g(s,s') = 0$, there can not be only one bond per any site. As in the Ising case considered in lectures, each bond can only be considered once, and the only graphs that survive have no dangling bonds. As a result, for a one-dimensional chain, with for instance open boundary conditions, it is impossible to draw any acceptable graph, and we obtain

$$Z = \sum_{\{s_i\}} \prod_{\langle ij \rangle} C(K)\left[1 + T(K)g(s_i,s_j)\right] = C(K)^{N-1}q^N = q\left(e^K + q - 1\right)^{N-1}.$$

Ignoring the boundary effects, i.e., that there are $N-1$ bonds in the chain, the free energy per site is obtained as

$$-\frac{\beta F}{N} = \ln\left(e^K + q - 1\right).$$

With the same method, we can also calculate the correlation function $\langle g(s_n s_m)\rangle$. To get a nonzero contribution, we have to consider a graph that directly connects these two sites. Assuming that $n > m$, this gives

$$\langle g(s_n s_m)\rangle = \frac{C(K)^N}{Z}\sum_{\{s_i\}} g(s_n s_m)\prod_{\langle ij\rangle}\left[1+T(K)g(s_i,s_j)\right]$$

$$= \frac{C(K)^N}{Z}T(K)^{n-m}\sum_{\{s_i\}} g(s_n s_m)g(s_m,s_{m+1})\cdots g(s_{n-1},s_n)$$

$$= \frac{C(K)^N}{Z}T(K)^{n-m}q^{n-m+1}(q-1)q^{N-(n-m)-1} = T^{n-m}(q-1)$$

where we have used the relationships obtained in (b).

(d) Calculate the partition function on the square lattice to order of T^4. Also calculate the first term in the low–temperature expansion of this problem.

• The first term in the high temperature series for a square lattice comes from a square of 4 bonds. There are a total of N such squares. Therefore,

$$Z = \sum_{\{s_i\}}\prod_{\langle ij\rangle} C(K)\left[1+T(K)g(s_i,s_j)\right] = C(K)^{2N}q^N\left[1+NT(K)^4(q-1)+\cdots\right].$$

Note that any closed loop involving ℓ bonds without intersections contributes $T^\ell q^\ell(q-1)$.

On the other hand, at low temperatures, the energy is minimized by the spins all being in one of the q possible states. The lowest energy excitation is a single spin in a different state, resulting in an energy cost of $K\times 4$ with a degeneracy factor $N\times(q-1)$, resulting in

$$Z = qe^{2NK}\left[1+N(q-1)e^{-4K}+\cdots\right].$$

(e) By comparing the first terms in low- and high–temperature series, find a duality rule for Potts models. Don't worry about higher order graphs, they will work out! Assuming a single transition temperature, find the value of $K_c(q)$.

• Comparing these expansions, we find the following duality condition for the Potts model

$$e^{-\tilde{K}} = T(K) = \frac{e^K-1}{e^K+q-1}.$$

This duality rule maps the low temperature expansion to a high temperature series, or vice versa. It also maps pairs of points, $\tilde{K}\Leftrightarrow K$, since we can rewrite the above relationship in a symmetric way

$$\left(e^{\tilde{K}}-1\right)\left(e^K-1\right) = q,$$

and consequently, if there is a single singular point K_c, it must be self-dual point,

$$K_c = \tilde{K}_c, \qquad \Longrightarrow \qquad K_c = \ln\left(\sqrt{q}+1\right).$$

(f) How do the higher order terms in the high–temperature series for the Potts model differ from those of the Ising model? What is the fundamental difference that sets apart the graphs for $q = 2$? (This is ultimately the reason why only the Ising model is solvable.)

• As mentioned in lectures, the Potts model with $q = 2$ can be mapped to the Ising model by noticing that $\delta_{ss'} = (1 + ss')/2$. However, higher order terms in the high-temperature series of the Potts model involve, in general, graphs with *three* or more bonds emanating from each site. These configurations do not correspond to a random walk, not even a constrained one as introduced in class for the 2d-Ising model on a square lattice. The quantity

$$\sum_{s_1=1}^{q} g(s_1, s_2)g(s_1, s_3)g(s_1, s_4) = q^3\delta_{s_2s_3}\delta_{s_2s_4} - q^2(\delta_{s_2s_3} + \delta_{s_2s_4} + \delta_{s_3s_4}) + 2q,$$

is always zero when $q = 2$ (as can be easily checked for any possible state of the spins s_2, s_3 and s_4), but is in general different from zero for $q > 2$. This is the fundamental difference that ultimately sets apart the case $q = 2$. Note that the corresponding diagrams in the low temperature expansion involve adjacent regions in 3 (or more) distinct states.

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3. *Potts model II:* An alternative expansion is obtained by starting with

$$\exp\left[K\delta(s_i, s_j)\right] = 1 + v(K)\delta(s_i, s_j),$$

where $v(K) = e^K - 1$. In this case, the sum over spins *does not* remove any graphs, and all choices of distributing bonds at random on the lattice are acceptable.

(a) Including a magnetic field $h\sum_i \delta_{s_i,1}$, show that the partition function takes the form

$$Z(q, K, h) = \sum_{\text{all graphs}} \prod_{\text{clusters c in graph}} \left[v^{n_b^c} \times \left(q - 1 + e^{hn_s^c}\right)\right],$$

where n_b^c and n_s^c are the numbers of bonds and sites in cluster c. This is known as the *random cluster expansion.*

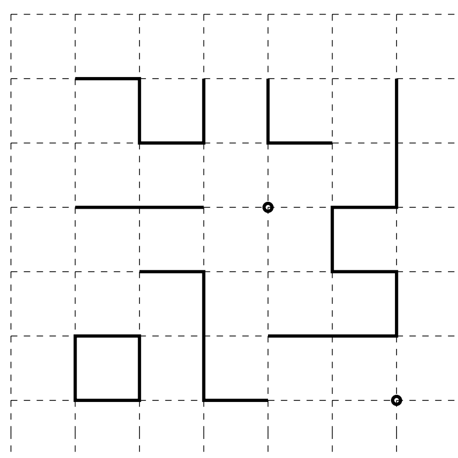
• Including a symmetry breaking fiela along direction 1, the partition function

$$Z = \sum_{\{s_i\}} \prod_{\langle ij\rangle} \left[1 + v(K)\delta(s_i, s_j)\right] \prod_i e^{h\delta_{s_i,1}},$$

can be expanded in powers of $v(K)$ as follows. As usual, if there is a total number N_B of nearest neighbor bonds on the lattice, the product over bonds generates 2^{N_B} possible terms. Each term may be represented by a graph on the lattice, in which a bond joining sites i and j is included if the factor $v\delta(s_i, s_j)$ appears in the term considered. Each included bond carries a factor $v(K)$, as well as a delta function enforcing the equality of the spins on the sites which it connects. In general, these bonds form clusters of different sizes and shapes, and within each cluster, the delta functions force the spins at each vertex to be the same. The sum $\sum_{\{s_i\}}$ therefore gives a factor of $(q-1)+e^{hn_s^c}$ for each cluster c, where n_s^c is the number of point in the cluster. The partition function may then be written as

$$Z(q,v,h) = \sum_{\text{all graphs}} \prod_{\text{clusters in graph}} \left[v(K)^{n_b^c} \left(q - 1 + e^{hn_c^s} \right) \right],$$

where n_b^c is the number of bonds in cluster c, and the sum runs over all distinct cluster collections. Note that an isolated site is also included in this definition of a cluster. While the Potts model was originally defined for integer q, using this expansion, we can evaluate Z for all values of q.



(b) Show that the limit $q \to 1$ describes a *percolation* problem, in which bonds are randomly distributed on the lattice with probability $p = v/(v+1)$. What is the percolation threshold on the square lattice?

• In the problem of *bond percolation*, bonds are independently distributed on the lattice, with a probability p of being present. The weight for a given configuration of occupied and absent bonds bonds is therefore

$$W(\text{graph}) = (1-p)^{zN} \prod_{\text{clusters in graph}} \left(\frac{p}{1-p} \right)^{n_b^c}.$$

The prefactor of $(1-p)^{zN}$ is merely the weight of the configuration with no bonds. The above weights clearly become identical to those appearing in the random cluster expansion of the Potts model for $q = 1$ (and $h = 0$). Clearly, we have to set $p = v/(v + 1)$, and neglect an overall factor of $(1+v)^N$, which is analytic in v, and does not affect any singular behavior. The partition function itself is trivial in this limit as $Z(1, v, h) = (1 + v)^{zN} e^{hN}$. On the other hand, we can obtain information on the number of clusters by considering the limt of $q \to 1$ from

$$\left.\frac{\partial \ln Z(q, v)}{\partial q}\right|_{q=1} = \sum_{\text{all graphs}} [\text{probablility of graph}] \sum_{\text{clusters in graph}} e^{-hn_c^s}.$$

Various properties of interest to percolation can then be calculated from the above generating function. This mapping enables us to extract the scaling laws at the percolation point, which is a continuous geometrical phase transition. The analog of the critical temperature is played by the percolation threshold p_c, which we can calculate using duality as $p_c = 1/2$ (after noting that $v^* = 1$).

An alternative way of obtaining this threshold is to find a duality rule for the percolation problem itself: One can similarly think of the problem in terms of empty bonds with a corresponding probability q. As p plays the role of temperature, there is a mapping of low p to high q or vice versa, and such that $q = 1 - p$. The self-dual point is then obtained by setting $p^* = 1 - p^*$, resulting in $p^* = 1/2$.

(c) Show that in the limit $q \to 0$, only a single connected cluster contributes to leading order. The enumeration of all such clusters is known as listing *branched lattice animals.*

• The parition function $Z(q, v, h)$ goes to zero at $q = 0$, but again infomration about geometrical lattice structure can be obtain by taking the limit $q \to 0$ in an appropriate fashion. In particular, if we set $v = q^a x$, then

$$Z(q, v = xq^a, h = 0) = \sum_{\text{all graphs}} x^{N_b} q^{N_c + aN_b},$$

where N_b and N_c are the total number of bonds and clusters. The leading dependence on q as $q \to 0$ comes from graphs with the lowest number of $N_c + aN_b$, and depends on the value of a. For $0 < a < 1$, these are the *spanning trees*, which connect all sites of the lattice (hence $N_c = 1$) and that enclose no loops (hence $N_b = N - 1$). Such spanning trees have a power of $x^{a(N-1)} q^{aN-a+1}$, and all other graphs have higher powers of q. For $a = 0$ one can add bonds to the spanning cluster (creating loops) without changing the power, as long as all sites remain connected in a single cluster. These have a relation to a problem referred to as *branched lattice animals.*

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4. *Potts duality:* Consider Potts spins, $s_i = (1, 2, \cdots, q)$, placed on the sites of a *square lattice* of N sites, interacting with their nearest-neighbors through a Hamiltonian

$$-\beta\mathcal{H} = K \sum_{<ij>} \delta_{s_i, s_j}.$$

(a) By comparing the first terms of high and low temperature series, or by any other method, show that the partition function has the property

$$Z(K) = qe^{2NK}\Xi\left[e^{-K}\right] = q^{-N}\left[e^K + q - 1\right]^{2N}\Xi\left[\frac{e^K - 1}{e^K + (q-1)}\right],$$

for some function Ξ, and hence locate the critical point $K_c(q)$.

• The low temperature series takes the form

$$Z = qe^{2NK}\left[1 + N(q-1)e^{-4K} + \cdots\right] \equiv qe^{2NK}\Xi\left[e^{-K}\right],$$

while at high temperatures

$$\begin{aligned} Z &= \left[\frac{e^K + q - 1}{q}\right]^{2N} q^N \left[1 + N(q-1)\left(\frac{e^K - 1}{e^K + q - 1}\right)^4 + \cdots\right] \\ &\equiv q^{-N}\left[e^K + q - 1\right]^{2N}\Xi\left[\frac{e^K - 1}{e^K + q - 1}\right]. \end{aligned}$$

Both of the above series for Ξ are in fact the same, leading to the duality condition

$$e^{-\tilde{K}} = \frac{e^K - 1}{e^K + q - 1},$$

and a critical (self-dual) point of

$$K_c = \tilde{K}_c, \qquad \Longrightarrow \qquad K_c = \ln\left(\sqrt{q} + 1\right).$$

(b) Starting from the duality expression for $Z(K)$, derive a similar relation for the internal energy $U(K) = \langle \beta\mathcal{H} \rangle = -\partial \ln Z/\partial \ln K$. Use this to calculate the exact value of U at the critical point.

• The duality relation for the partition function gives

$$\ln Z(K) = \ln q + 2NK + \ln\Xi\left[e^{-K}\right] = -N\ln q + 2N\ln\left[e^K + q - 1\right] + \ln\Xi\left[\frac{e^K - 1}{e^K + q - 1}\right].$$

The internal energy $U(K)$ is then obtained from

$$\begin{aligned} -\frac{U(K)}{K} = \frac{\partial}{\partial K}\ln Z(K) &= 2N - e^{-K}\ln\Xi'\left[e^{-K}\right] \\ &= 2N\frac{e^K}{e^K + q - 1} + \frac{qe^K}{\left(e^K + q - 1\right)^2}\ln\Xi'\left[\frac{e^K - 1}{e^K + q - 1}\right]. \end{aligned}$$

$\ln \Xi'$ is the derivative of $\ln \Xi$ with respect to its argument, whose value is not known in general. However, at the critical point K_c, the arguments of $\ln \Xi'$ from the high and low temperature forms of the above expression are the same. Substituting $e^{K_c} = 1 + \sqrt{q}$, we obtain

$$2N - \frac{\ln \Xi'_c}{1+\sqrt{q}} = \frac{2N}{\sqrt{q}} + \frac{\ln \Xi'_c}{1+\sqrt{q}}, \quad \Longrightarrow \quad \ln \Xi'_c = \frac{q-1}{\sqrt{q}} N,$$

and,

$$-\frac{U(K_c)}{K_c} = N\left(2 - \frac{q-1}{\sqrt{q}+q}\right), \quad \Longrightarrow \quad U(K_c) = NK_c \frac{\sqrt{q}+1}{\sqrt{q}}.$$

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5. *Anisotropic Random Walks:* Consider the ensemble of all random walks on a square lattice starting at the origin (0,0). Each walk has a weight of ${t_x}^{\ell_x} \times {t_y}^{\ell_y}$, where ℓ_x and ℓ_y are the number of steps taken along the x and y directions respectively.

(a) Calculate the total weight $W(x,y)$, of all walks terminating at (x,y). Show that W is well defined only for $\bar{t} = (t_x + t_y)/2 < t_c = 1/4$.

• Defining $\langle 0,0|\, W(\ell)\, |x,y\rangle$ to be the weight of all walks of ℓ steps terminating at (x,y), we can follow the steps in sec.VI.F of the lecture notes. In the anisotropic case, Eq.(VI.47) (applied ℓ times) is trivially recast into

$$\begin{aligned}\langle x,y|\, T^\ell\, |q_x,q_y\rangle &= \sum_{x',y'} \langle x,y|\, T^\ell\, |x',y'\rangle \langle x',y'|\, q_x,q_y\rangle \\ &= \left(2t_x \cos q_x + 2t_y \cos q_y\right)^\ell \langle x,y|\, q_x,q_y\rangle\,,\end{aligned}$$

where $\langle x,y|\, q_x,q_y\rangle = e^{iq_x x + iq_y y}/\sqrt{N}$. Since $W(x,y) = \sum_\ell \langle 0,0|\, W(\ell)\, |x,y\rangle$, its Fourier transform is calculated as

$$\begin{aligned}W(q_x,q_y) &= \sum_\ell \sum_{x,y} \langle 0,0|\, T^\ell\, |x,y\rangle \langle x,y|\, q_x,q_y\rangle \\ &= \sum_\ell \left(2t_x \cos q_x + 2t_y \cos q_y\right)^\ell = \frac{1}{1 - \left(2t_x \cos q_x + 2t_y \cos q_y\right)}.\end{aligned}$$

Finally, Fourier transforming back gives

$$W(x,y) = \int_{-\pi}^{\pi} \frac{d^2q}{(2\pi)^2} W(q_x,q_y)\, e^{-iq_x x - iq_y y} = \int_{-\pi}^{\pi} \frac{d^2q}{(2\pi)^2} \frac{e^{-iq_x x - iq_y y}}{1 - \left(2t_x \cos q_x + 2t_y \cos q_y\right)}.$$

Note that the summation of the series is legitimate (for all q's) only for $2t_x + 2t_y < 1$, *i.e.* for $\bar{t} = (t_x + t_y)/2 < t_c = 1/4$.

(b) What is the shape of a curve $W(x,y) =$ constant, for large x and y, and close to the transition?

• For x and y large, the main contributions to the above integral come from small q's. To second order in q_x and q_y, the denominator of the integrand reads

$$1 - 2\left(t_x + t_y\right) + t_x q_x^2 + t_y q_y^2.$$

Then, with $q_i' \equiv \sqrt{t_i} q_i$, we have

$$W\left(x, y\right) \approx \int_{-\infty}^{\infty} \frac{d^2 q'}{\left(2\pi\right)^2 \sqrt{t_x t_y}} \frac{e^{-i\mathbf{q}'\cdot\mathbf{v}}}{1 - 2\left(t_x + t_y\right) + \mathbf{q}'^2},$$

where we have extended the limits of integration to infinity, and $\mathbf{v} = \left(\frac{x}{\sqrt{t_x}}, \frac{y}{\sqrt{t_y}}\right)$. As the denominator is rotationally invariant, the integral depends only on the magnitude of the vector $\mathbf{v}$. In other words, $W\left(x, y\right)$ is constant along ellipses

$$\frac{x^2}{t_x} + \frac{y^2}{t_y} = \text{constant}.$$

(c) How does the average number of steps, $\langle \ell \rangle = \langle \ell_x + \ell_y \rangle$, diverge as $\bar{t}$ approaches t_c?

• The weight of all walks of length ℓ, irrespective of their end point location, is

$$\sum_{x,y} \langle 0, 0| W\left(\ell\right) |x, y\rangle = \langle 0, 0| T^\ell |q_x = 0, q_y = 0\rangle = \left(2t_x + 2t_y\right)^\ell = \left(4\bar{t}\,\right)^\ell.$$

Therefore,

$$\langle \ell \rangle = \frac{\sum_\ell \ell \left(4\bar{t}\,\right)^\ell}{\sum_\ell \left(4\bar{t}\,\right)^\ell} = 4\bar{t} \frac{\partial}{\partial\left(4\bar{t}\,\right)} \ln\left[\sum_\ell \left(4\bar{t}\,\right)^\ell\right] = 4\bar{t} \frac{\partial}{\partial\left(4\bar{t}\,\right)} \ln \frac{1}{1 - 4\bar{t}} = \frac{4\bar{t}}{1 - 4\bar{t}},$$

i.e.

$$\langle \ell \rangle = \frac{\bar{t}}{t_c - \bar{t}},$$

diverges linearly close to the singular value of $\bar{t}$.

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6. *Anisotropic Ising Model:* Consider the anisotropic Ising model on a square lattice with a Hamiltonian

$$-\beta\mathcal{H} = \sum_{x,y} \left(K_x \sigma_{x,y}\sigma_{x+1,y} + K_y \sigma_{x,y}\sigma_{x,y+1}\right);$$

i.e. with bonds of different strengths along the x and y directions.

(a) By following the method presented in the text, calculate the free energy for this model. You do not have to write down every step of the derivation. Just sketch the steps that need to be modified due to anisotropy; and calculate the final answer for $\ln Z/N$.

• The Hamiltonian

$$-\beta\mathcal{H}=\sum_{x,y}\left(K_x\sigma_{x,y}\sigma_{x+1,y}+K_y\sigma_{x,y}\sigma_{x,y+1}\right),$$

leads to

$$Z=\sum\left(2\cosh K_x\cosh K_y\right)^N t_x^{\ell_x}t_y^{\ell_y},$$

where $t_i=\tanh K_i$, and the sum runs over all closed graphs. By extension of the isotropic case,

$$f=\frac{\ln Z}{N}=\ln\left(2\cosh K_x\cosh K_y\right)+\sum_{\ell_x,\ell_y}\frac{t_x^{\ell_x}t_y^{\ell_y}}{\ell_x+\ell_y}\left\langle 0\right|W^*\left(\ell_x,\ell_y\right)\left|0\right\rangle,$$

where

$$\left\langle 0\right|W^*\left(\ell_x,\ell_y\right)\left|0\right\rangle=\frac{1}{2}\sum{}^{\prime}(-1)^{\text{number of crossings}},$$

and the primed sum runs over all directed (ℓ_x,ℓ_y)-steps walks from $(0,0)$ to $(0,0)$ with no U-turns. As in the isotropic case, this is evaluated by taking the trace of powers of the $4N\times 4N$ matrix described by Eq.(VI.66), which is block-diagonalized by Fourier transformation. However, unlike the isotropic case, in which each element is multiplied by t, here they are multiplied by t_x and t_y, respectively, resulting in

$$f=\ln\left(2\cosh K_x\cosh K_y\right)+\frac{1}{2}\int\frac{d^2q}{(2\pi)^2}\mathrm{tr}\ln\left[1-\mathrm{T}\left(\mathbf{q}\right)^*\right],$$

where

$$\begin{aligned}\mathrm{tr}\ln\left[1-\mathrm{T}\left(\mathbf{q}\right)^*\right]&=\ln\det\left[1-T\left(\mathbf{q}\right)^*\right]\\&=\ln\left[\left(1+t_x^2\right)\left(1+t_y^2\right)-2t_x\left(1-t_y^2\right)\cos q_x-2t_y\left(1-t_x^2\right)\cos q_y\right]\\&=\ln\left[\frac{\cosh 2K_x\cosh 2K_y-\sinh 2K_x\cos q_x-\sinh 2K_y\cos q_y}{\cosh^2 K_x\cosh^2 K_y}\right],\end{aligned}$$

resulting in

$$f=\ln 2+\frac{1}{2}\int\frac{d^2q}{(2\pi)^2}\ln\left(\cosh 2K_x\cosh 2K_y-\sinh 2K_x\cos q_x-\sinh 2K_y\cos q_y\right).$$

(b) Find the critical boundary in the (K_x,K_y) plane from the singularity of the free energy. Show that it coincides with the condition $K_x=\tilde{K}_y$, where $\tilde{K}$ indicates the standard dual interaction to K.

• The argument of the logarithm is minimal at $q_x=q_y=0$, and equal to

$$\begin{aligned}&\cosh 2K_x\cosh 2K_y-\sinh 2K_x-\sinh 2K_y\\&=\frac{1}{2}\left(e^{K_x}\sqrt{\cosh 2K_y-1}-e^{-K_x}\sqrt{\cosh 2K_y+1}\right)^2.\end{aligned}$$

Therefore, the critical line is given by

$$e^{2K_x} = \sqrt{\frac{\cosh 2K_y + 1}{\cosh 2K_y - 1}} = \coth K_y.$$

Note that this condition can be rewritten as

$$\sinh 2K_x = \frac{1}{2}\left(\coth K_y - \tanh K_y\right) = \frac{1}{\sinh 2K_y},$$

i.e. the critical boundary can be described as $K_x = \tilde{K}_y$, where the dual interactions, $\tilde{K}$ and K, are related by $\sinh 2K \sinh 2\tilde{K} = 1$.

(c) Find the singular part of $\ln Z/N$, and comment on how anisotropy affects critical behavior in the exponent and amplitude ratios.

• The singular part of $\ln Z/N$ for the anisotropic case can be written as

$$f_S = \frac{1}{2}\int \frac{d^2q}{(2\pi)^2} \ln\left[\left(e^{K_x}\sqrt{\cosh 2K_y - 1} - e^{-K_x}\sqrt{\cosh 2K_y + 1}\right)^2 + \sum_{i=x,y} \frac{q_i^2}{2}\sinh 2K_i\right].$$

In order to rewrite this expression in a form closer to that of the singular part of the free energy in the isotropic case, let

$$q_i = \sqrt{\frac{2}{\sinh 2K_i}} q_i',$$

and

$$\delta t = e^{K_x}\sqrt{\cosh 2K_y - 1} - e^{-K_x}\sqrt{\cosh 2K_y + 1}$$

(δt goes linearly through zero as (K_x, K_y) follows a curve which intersects the critical boundary). Then

$$f_S = \frac{1}{\sqrt{\sinh 2K_x \sinh 2K_y}} \int \frac{d^2q'}{(2\pi)^2} \ln\left(\delta t^2 + q'^2\right).$$

Thus, upon approaching the critical boundary ($\sinh 2K_x \sinh 2K_y = 1$), the singular part of the anisotropic free energy coincides more and more precisely with the isotropic one, and the exponents and amplitude ratios are unchanged by the anisotropy. (The amplitudes themselves obviously depend on the locatio

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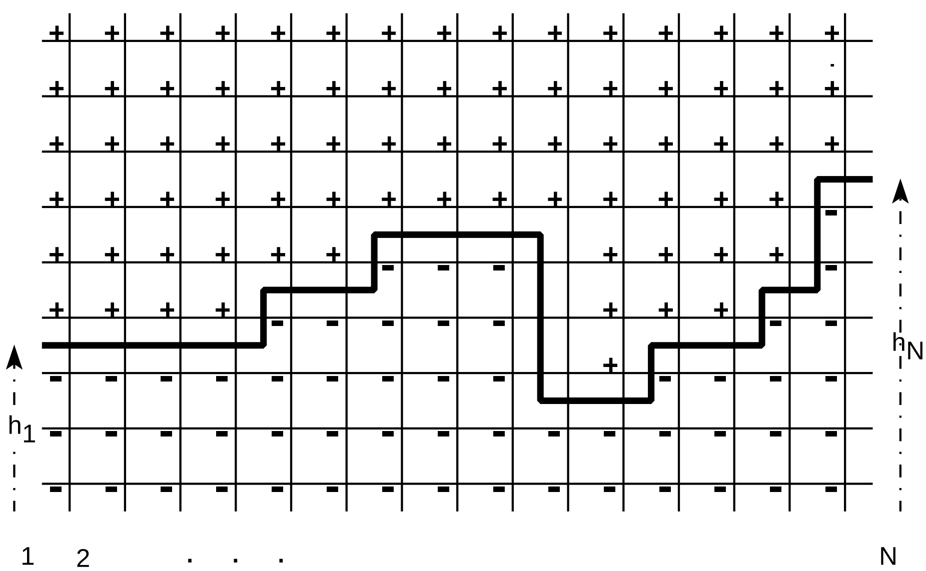

7. *Müller–Hartmann Zittartz estimate* of the interfacial energy of the $d = 2$ Ising model on a square lattice:

(a) Consider an interface on the square lattice with periodic boundary conditions in one direction. Ignoring islands and overhangs, the configurations can be labelled by heights h_n for $1 \leq n \leq L$. Show that for an ansiotropic Ising model of interactions (K_x, K_Y), the energy of an interface along the x-direction is

$$-\beta\mathcal{H} = -2K_y L - 2K_x \sum_n |h_{n+1} - h_n| \; .$$

• For each unsatisfied $(+-)$ bond, the energy is increased by $2K_i$ from the ground state energy, with $i = x$ if the unsatisfied bond is vertical, and $i = y$ if the latter is horizontal. Ignoring islands and overhangs, the number of horizontal bond of the interface is L, while the number of vertical bonds is $\sum_n |h_{n+1} - h_n|$, yielding

$$-\beta\mathcal{H} = -2K_y L - 2K_x \sum_{n=1}^{L} |h_{n+1} - h_n| \, .$$

(b) Write down a column–to–column transfer matrix $\langle h|T|h'\rangle$, and diagonalize it.

• We can define

$$\langle h| \, T \, |h'\rangle \equiv \exp\left(-2K_y - 2K_x \left|h' - h\right|\right),$$

or, in matrix form,

$$T = e^{-2K_y} \begin{pmatrix} 1 & e^{-2K_x} & e^{-4K_x} & \cdots & e^{-HK_x} & e^{-HK_x} & e^{-2\left(\frac{H}{2}-1\right)K_x} & \cdots & e^{-2K_x} \\ e^{-2K_x} & 1 & e^{-2K_x} & \cdots & e^{-2\left(\frac{H}{2}-1\right)K_x} & e^{-2\left(\frac{H}{2}+1\right)K_x} & e^{-HK_x} & \cdots & e^{-4K_x} \\ \cdots & & & & & & & & \end{pmatrix}$$

where H is the vertical size of the lattice. In the $H \to \infty$ limit, T is easily diagonalized since each line can be obtained from the previous line by a single column shift. The eigenvectors of such matrices are composed by the complex roots of unity (this is equivalent to the statement that a translationally invariant system is diagonal in Fourier modes). To the eigenvector

$$\left(e^{i\frac{2\pi}{k}}, e^{i\frac{2\pi}{k}\cdot 2}, e^{i\frac{2\pi}{k}\cdot 3}, \cdots, e^{i\frac{2\pi}{k}\cdot (H+1)}\right),$$

is associated the eigenvalue

$$\lambda_k = e^{-2K_y} \sum_{n=1}^{H+1} T_{1n} e^{i\frac{2\pi}{k}\cdot(n-1)}.$$

Note that there are $H+1$ eigenvectors, corresponding to $k = 1, \cdots, H+1$.

(c) Obtain the interface free energy using the result in (b), or by any other method.

• One way of obtaining the free energy is to evaluate the largest eigenvalue of T. Since all elements of T are positive, the eigenvector $(1, 1, \cdots, 1)$ has the largest eigenvalue

$$\begin{aligned}\lambda_1 &= e^{-2K_y} \sum_{n=1}^{H+1} T_{1n} = e^{-2K_y} \left(1 + 2\sum_{n=1}^{H/2} e^{-2K_x n}\right) \\ &= e^{-2K_y} \left(2\sum_{n=0}^{H/2} e^{-2K_x n} - 1\right) = e^{-2K_y} \coth K_x,\end{aligned}$$

in the $H \to \infty$ limit. Then, $F = -Lk_BT \ln \lambda_1$.

Alternatively, we can directly sum the partition function, as

$$\begin{aligned}Z &= e^{-2K_yL} \sum_{\{h_n\}} \exp\left(-2K_x \sum_{n=1}^{L} |h_{n+1} - h_n|\right) = e^{-2K_yL} \left[\sum_d \exp\left(-2K_x\,|d|\right)\right]^L \\ &= \left[e^{-2K_y}\left(2\sum_{d\geq 0} e^{-2K_x d} - 1\right)\right]^L = \left(e^{-2K_y} \coth K_x\right)^L,\end{aligned}$$

yielding

$$F = -Lk_BT \left[\ln\left(\coth K_x\right) - 2K_y\right].$$

(d) Find the condition between K_x and K_y for which the interfacial free energy vanishes. Does this correspond to the critical boundary of the original 2d Ising model?

• The interfacial free energy vanishes for

$$\coth K_x = e^{2K_y},$$

which coincides with the result from an earlier problem. This illustrates that long wavelength fluctuations, such as interfaces, are responsible for destroying order at criticality.

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8. *Anisotropic Landau Theory:* Consider an n–component magnetization field $\vec{m}(\mathbf{x})$ in d–dimensions.

(a) Using the previous problems on anisotropy as a guide, generalize the standard Landau–Ginzburg Hamiltonian to include the effects of spacial anisotropy.

• Requiring different coupling constants in the different spatial directions, along with rotational invariance in spin space, leads to the following leading terms of the Hamiltonian,

$$-\beta\mathcal{H} = \int d^d x \left[\frac{t}{2} \vec{m}\left(\mathbf{x}\right)^2 + \sum_{i=1}^{d} \frac{K_i}{2} \frac{\partial \vec{m}}{\partial x_i} \cdot \frac{\partial \vec{m}}{\partial x_i} + u \vec{m}\left(\mathbf{x}\right)^4 \right].$$

(b) Are such anisotropies "relevant?"

• Clearly, the apparent anisotropy can be eliminated by the rescaling

$$x_i' = \sqrt{\frac{K}{K_i}} x_i.$$

In terms of the primed space variables, the Hamiltonian is isotropic. In particular, the universal features are identical in the anisotropic and isotropic cases, and the anisotropy is thus "irrelevant" (provided all K_i are non-vanishing).

(c) In La_2CuO_4, the Cu atoms are arranged on the sites of a square lattice in planes, and the planes are then stacked together. Each Cu atom carries a "spin", which we assume to be classical, and can point along any direction in space. There is a very strong antiferromagnetic interaction in each plane. There is also a very weak interplane interaction that prefers to align successive layers. Sketch the low–temperature magnetic phase, and indicate to what universality class the order–disorder transition belongs.

• For classical spins, this combination of antiferromagnetic and ferromagnetic couplings is equivalent to a purely ferromagnetic (anisotropic) system, since we can redefine (*e.g.* in the partition function) all the spins on one of the two sublattices with an opposite sign. Therefore, the critical behavior belongs to the $d = 3$, $n = 3$ universality class.

Nevertheless, there is a range of temperatures for which the in-plane correlation length is large compared to the lattice spacing, while the interplane correlation length is of the order of the lattice spacing. The behavior of the system is then well described by a $d = 2$, $n = 3$ theory.

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9. *Anisotropic nonlinear σ model:* Consider unit n-component spins, $\vec{s}(\mathbf{x}) = (s_1, \cdots, s_n)$ with $\sum_\alpha s_\alpha^2 = 1$, subject to a Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{1}{2T}\left(\nabla\vec{s}\right)^2 + g s_1^2\right].$$

For $g = 0$, renormalization group equations are obtained through rescaling distances by a factor $b = e^\ell$, and spins by a factor $\zeta = b^{y_s}$ with $y_s = -\frac{(n-1)}{4\pi}T$, and lead to the flow equation

$$\frac{dT}{d\ell} = -\epsilon T + \frac{(n-2)}{2\pi}T^2 + \mathcal{O}(T^3),$$

where $\epsilon = d - 2$.

(a) Find the fixed point, and the thermal eigenvalue y_T.

• Setting $dT/d\ell$ to zero, the fixed point is obtained as

$$T^* = \frac{2\pi\epsilon}{n-2} + \mathcal{O}(\epsilon^2).$$

Linearizing the recursion relation gives

$$y_T = -\epsilon + \frac{(n-2)}{\pi}T^* = +\epsilon + \mathcal{O}(\epsilon^2).$$

(b) Write the renormalization group equation for g in the vicinity of the above fixed point, and obtain the corresponding eigenvalue y_g.

• Rescalings $x \to b\mathbf{x}'$ and $\vec{s} \to \zeta\vec{s}\,'$, lead to $g \to g' = b^d\zeta^2 g$, and hence

$$y_g = d + 2y_s = d - \frac{n-1}{2\pi}T^* = 2 + \epsilon - \frac{n-1}{n-2}\epsilon = 2 - \frac{1}{n-2}\epsilon + \mathcal{O}(\epsilon^2).$$

(c) Sketch the phase diagram as a function of T and g, indicating the phases, and paying careful attention to the shape of the phase boundary as $g \to 0$.

•

The term proportional to g removes full rotational symmetry and leads to a bicritical phase diagram as discussed in recitations. The phase for $g < 0$ has order along direction 1, while $g > 0$ favors ordering along any one of the $(n-1)$ directions orthogonal to 1. The phase boundaries as $g \to 0$ behave as $g \propto (\delta T)^\phi$, with $\phi = y_g/y_t \approx 2/\epsilon + \mathcal{O}(1)$.

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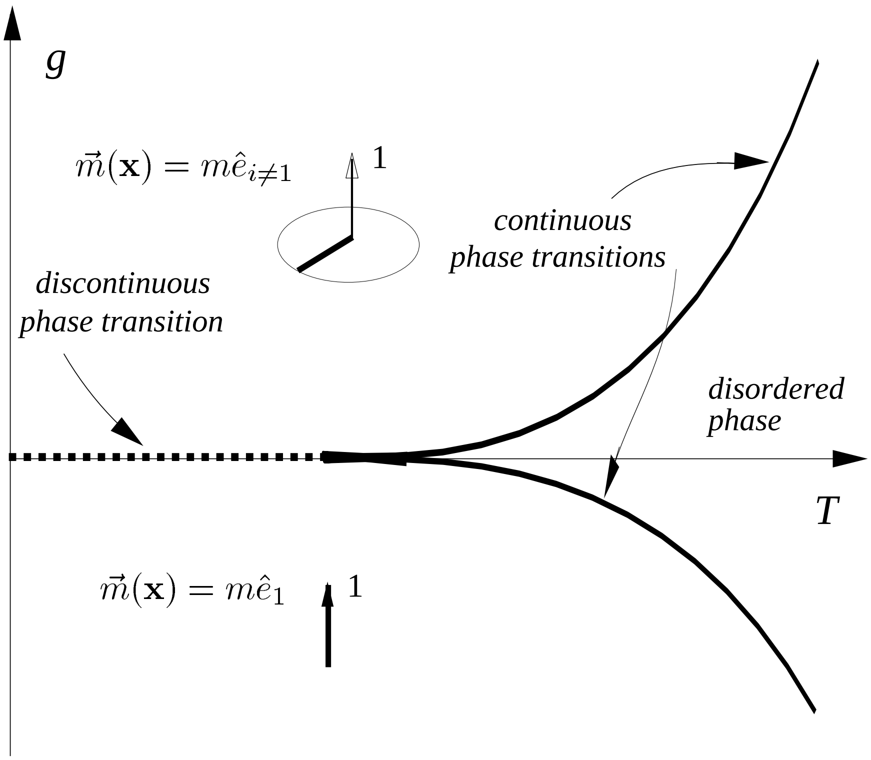
10. *Matrix models:* In some situations, the order parameter is a matrix rather than a vector. For example, in triangular (Heisenberg) antiferromagnets each triplet of spins aligns at 120°, locally defining a plane. The variations of this plane across the system are



described by a 3×3 rotation matrix. We can construct a nonlinear σ model to describe a generalization of this problem as follows. Consider the Hamiltonian

$$\beta\mathcal{H} = \frac{K}{4} \int d^d\mathbf{x} \ \mathrm{tr} \left[\nabla M(\mathbf{x}) \cdot \nabla M^T(\mathbf{x}) \right] \quad ,$$

where M is a *real,* $N \times N$ *orthogonal* matrix, and 'tr' denotes the trace operation. The condition of orthogonality is that $MM^T = M^T M = I$, where I is the $N \times N$ identity matrix, and M^T is the transposed matrix, $M^T_{ij} = M_{ji}$. The partition function is obtained by summing over all matrix functionals, as

$$Z = \int \mathcal{D}M(\mathbf{x}) \delta \left(M(\mathbf{x}) M^T(\mathbf{x}) - I \right) e^{-\beta\mathcal{H}[M(\mathbf{x})]} \quad .$$

(a) Rewrite the Hamiltonian and the orthogonality constraint in terms of the matrix elements M_{ij} $(i, j = 1, \cdots, N)$. Describe the ground state of the system.

• In terms of the matrix elements, the Hamiltonian reads

$$\beta\mathcal{H} = \frac{K}{4} \int d^d x \sum_{i,j} \nabla M_{ij} \cdot \nabla M_{ij},$$

and the orthogonality condition becomes

$$\sum_k M_{ik} M_{jk} = \delta_{ij}.$$

Since $\nabla M_{ij} \cdot \nabla M_{ij} \geq 0$, any constant (spatially uniform) orthogonal matrix realizes a ground state.

(b) Define the symmetric and anti-symmetric matrices

$$\begin{cases} \sigma = \dfrac{1}{2}\left(M + M^T\right) = \sigma^T \\ \pi = \dfrac{1}{2}\left(M - M^T\right) = -\pi^T \end{cases}.$$

Express $\beta\mathcal{H}$ and the orthogonality constraint in terms of the matrices σ and π.

• As $M = \sigma + \pi$ and $M^T = \sigma - \pi$,

$$\beta\mathcal{H} = \frac{K}{4}\int d^d x \,\mathrm{tr}\left[\nabla\left(\sigma + \pi\right) \cdot \nabla\left(\sigma - \pi\right)\right] = \frac{K}{4}\int d^d x \,\mathrm{tr}\left[\left(\nabla\sigma\right)^2 - \left(\nabla\pi\right)^2\right],$$

where we have used the (easily checked) fact that the trace of the cummutator of matrices $\nabla\sigma$ and $\nabla\pi$ is zero. Similarly, the orthogonality condition is written as

$$\sigma^2 - \pi^2 = I,$$

where I is the unit matrix.

(c) Consider *small fluctuations* about the ordered state $M(\mathbf{x}) = I$. Show that σ can be expanded in powers of π as

$$\sigma = I - \frac{1}{2}\pi\pi^T + \cdots.$$

Use the orthogonality constraint to integrate out σ, and obtain an expression for βH to fourth order in π. Note that there are two distinct types of fourth order terms. *Do not include* terms generated by the argument of the delta function. As shown for the nonlinear σ model in the text, these terms do not effect the results at lowest order.

• Taking the square root of

$$\sigma^2 = I + \pi^2 = I - \pi\pi^T,$$

results in

$$\sigma = I - \frac{1}{2}\pi\pi^T + \mathcal{O}\left(\pi^4\right),$$

(as can easily be checked by calculating the square of $I - \pi\pi^T/2$). We now integrate out σ, to obtain

$$Z = \int \mathcal{D}\pi\left(\mathbf{x}\right)\exp\left\{\frac{K}{4}\int d^d x \,\mathrm{tr}\left[\left(\nabla\pi\right)^2 - \frac{1}{4}\left(\nabla\left(\pi\pi^T\right)\right)^2\right]\right\},$$

where $\mathcal{D}\pi\left(\mathbf{x}\right) = \prod_{j>i}\mathcal{D}\pi_{ij}\left(\mathbf{x}\right)$, and π is a matrix with zeros along the diagonal, and elements below the diagonal given by $\pi_{ij} = -\pi_{ji}$. Note that we have not included the terms generated by the argument of the delta function. Such term, which ensure that the measure of integration over π is symmetric, do not contribute to the renormalization of K at the lowest order. Note also that the fourth order terms are of two distinct types, due to the non-commutativity of π and $\nabla\pi$. Indeed,

$$\begin{aligned}\left[\nabla\left(\pi\pi^T\right)\right]^2 &= \left[\nabla\left(\pi^2\right)\right]^2 = \left[\left(\nabla\pi\right)\pi + \pi\nabla\pi\right]^2 \\ &= \left(\nabla\pi\right)\pi\cdot\left(\nabla\pi\right)\pi + \left(\nabla\pi\right)\pi^2\cdot\nabla\pi + \pi\left(\nabla\pi\right)^2\pi + \pi\left(\nabla\pi\right)\pi\cdot\nabla\pi,\end{aligned}$$

and, since the trace is unchanged by cyclic permutations,

$$\mathrm{tr}\left[\nabla\left(\pi\pi^T\right)\right]^2 = 2\,\mathrm{tr}\left[\left(\pi\nabla\pi\right)^2 + \pi^2\left(\nabla\pi\right)^2\right].$$

(d) For an N-vector order parameter there are $N-1$ Goldstone modes. Show that an orthogonal $N\times N$ order parameter leads to $N(N-1)/2$ such modes.

• The anti-symmetry of π imposes $N\left(N+1\right)/2$ conditions on the $N\times N$ matrix elements, and thus there are $N^2 - N\left(N+1\right)/2 = N\left(N-1\right)/2$ independent components (Goldstone modes) for the matrix. Alternatively, the orthogonality of M similarly imposes $N\left(N+1\right)/2$ constraints, leading to $N\left(N-1\right)/2$ degrees of freedom. [Note that in the analogous calculation for the $\mathcal{O}\left(n\right)$ model, there is one condition constraining the magnitude of the spins to unity; and the remaining $n-1$ angular components are Goldstone modes.]

(e) Consider the quadratic piece of $\beta\mathcal{H}$. Show that the two point correlation function in Fourier space is

$$\langle\pi_{ij}(\mathbf{q})\pi_{kl}(\mathbf{q}')\rangle = \frac{(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{Kq^2}\left[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right].$$

• In terms of the Fourier components $\pi_{ij}(\mathbf{q})$, the quadratic part of the Hamiltonian in (c) has the form

$$\beta\mathcal{H}_0 = \frac{K}{2}\sum_{i<j}\int\frac{d^d\mathbf{q}}{(2\pi)^d}q^2|\pi_{ij}(\mathbf{q})|^2,$$

leading to the bare expectation values

$$\langle\pi_{ij}\left(\mathbf{q}\right)\pi_{ij}\left(\mathbf{q}'\right)\rangle_0 = \frac{\left(2\pi\right)^d\delta^d\left(\mathbf{q}+\mathbf{q}'\right)}{Kq^2},$$

and

$$\langle\pi_{ij}\left(\mathbf{q}\right)\pi_{kl}\left(\mathbf{q}'\right)\rangle_0 = 0, \text{ if the pairs } (ij) \text{ and } (kl) \text{ are different.}$$

Furthermore, since π is anti-symmetric,

$$\langle\pi_{ij}\left(\mathbf{q}\right)\pi_{ji}\left(\mathbf{q}'\right)\rangle_0 = -\langle\pi_{ij}\left(\mathbf{q}\right)\pi_{ij}\left(\mathbf{q}'\right)\rangle_0,$$

and in particular $\langle \pi_{ii}(\mathbf{q})\,\pi_{jj}(\mathbf{q}')\rangle_0 = 0$. These results can be summarized by

$$\langle \pi_{ij}(\mathbf{q})\,\pi_{kl}(\mathbf{q}')\rangle_0 = \frac{(2\pi)^d\,\delta^d(\mathbf{q}+\mathbf{q}')}{Kq^2}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right).$$

We shall now construct a renormalization group by removing Fourier modes $M^>(\mathbf{q})$, with $\mathbf{q}$ in the shell $\Lambda/b < |\mathbf{q}| < \Lambda$.

(f) Calculate the coarse grained expectation value for $\langle \mathrm{tr}(\sigma)\rangle_0^>$ at low temperatures after removing these modes. Identify the scaling factor, $M'(\mathbf{x}') = M^<(\mathbf{x})/\zeta$, that restores $\mathrm{tr}(M') = \mathrm{tr}(\sigma') = N$.

• As a result of fluctuations of short wavelength modes, $\mathrm{tr}\,\sigma$ is reduced to

$$\begin{aligned}\langle \mathrm{tr}\,\sigma\rangle_0^> &= \left\langle \mathrm{tr}\left(I + \frac{\pi^2}{2} + \cdots\right)\right\rangle_0^> \approx N + \frac{1}{2}\left\langle \mathrm{tr}\,\pi^2\right\rangle_0^> \\ &= N + \frac{1}{2}\left\langle \sum_{i\neq j}\pi_{ij}\pi_{ji}\right\rangle_0^> = N - \frac{1}{2}\left\langle \sum_{i\neq j}\pi_{ij}^2\right\rangle_0^> = N - \frac{1}{2}\left(N^2 - N\right)\left\langle \pi_{ij}^2\right\rangle_0^> \\ &= N\left(1 - \frac{N-1}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{Kq^2}\right) = N\left[1 - \frac{N-1}{2K}I_d(b)\right].\end{aligned}$$

To restore $\mathrm{tr}\,M' = \mathrm{tr}\,\sigma' = N$, we rescale all components of the matrix by

$$\zeta = 1 - \frac{N-1}{2K}I_d(b).$$

NOTE: An orthogonal matrix M is invertible ($M^{-1} = M^T$), and therefore diagonalizable. In diagonal form, the transposed matrix is equal to the matrix itself, and so its square is the identity, implying that each eigenvalue is either $+1$ or -1. Thus, if M is chosen to be very close to the identity, all eigenvalues are $+1$, and $\mathrm{tr}\,M = N$ (as the trace is independent of the coordinate basis).

(g) Use perturbation theory to calculate the coarse grained coupling constant $\tilde{K}$. Evaluate only the two diagrams that directly renormalize the $(\nabla\pi_{ij})^2$ term in $\beta\mathcal{H}$, and show that

$$\tilde{K} = K + \frac{N}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{q^2} \quad .$$

• Distinguishing between the greater and lesser modes, we write the partition function as

$$Z = \int \mathcal{D}\pi^<\mathcal{D}\pi^>\,e^{-\beta\mathcal{H}_0^< - \beta\mathcal{H}_0^> + U[\pi^<,\pi^>]} = \int \mathcal{D}\pi^<\,e^{-\delta f_b^0 - \beta\mathcal{H}_0^<}\,\left\langle e^U\right\rangle_0^>,$$

where $\mathcal{H}_0$ denotes the quadratic part, and

$$U = -\frac{K}{8}\sum_{i,j,k,l}\int d^d x\left[\left(\nabla\pi_{ij}\right)\pi_{jk}\cdot\left(\nabla\pi_{kl}\right)\pi_{li} + \pi_{ij}\left(\nabla\pi_{jk}\right)\cdot\left(\nabla\pi_{kl}\right)\pi_{li}\right]$$

$$= \frac{K}{8}\sum_{i,j,k,l}\int\frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}}\left[\left(\mathbf{q}_1\cdot\mathbf{q}_3 + \mathbf{q}_2\cdot\mathbf{q}_3\right)\cdot\right.$$

$$\left.\cdot\pi_{ij}\left(\mathbf{q}_1\right)\pi_{jk}\left(\mathbf{q}_2\right)\pi_{kl}\left(\mathbf{q}_3\right)\pi_{li}\left(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3\right)\right].$$

To first order in U, the following two averages contribute to the renormalization of K:

$$(i)\qquad \frac{K}{8}\sum_{i,j,k,l}\int\frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}}\left\langle\pi^{>}_{jk}\left(\mathbf{q}_2\right)\pi^{>}_{li}\left(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3\right)\right\rangle^{>}_{0}\left(\mathbf{q}_1\cdot\mathbf{q}_3\right)\pi^{<}_{ij}\left(\mathbf{q}_1\right)\pi^{<}_{kl}\left(\mathbf{q}_3\right)$$

$$=\frac{K}{8}\left(\int_{\Lambda/b}^{\Lambda}\frac{d^d q'}{(2\pi)^d}\frac{1}{Kq'^2}\right)\left(\int_0^{\Lambda/b}\frac{d^d q}{(2\pi)^d}q^2\sum_{i,j}\pi^{<}_{ij}\left(\mathbf{q}\right)\pi^{<}_{ji}\left(-\mathbf{q}\right)\right),$$

and

$$(ii)\ \frac{K}{8}\sum_{j,k,l}\int\frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}}\left\langle\sum_{i\neq j,l}\pi^{>}_{ij}\left(\mathbf{q}_2\right)\pi^{>}_{li}\left(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3\right)\right\rangle^{>}_{0}\left(\mathbf{q}_2\cdot\mathbf{q}_3\right)\pi^{<}_{jk}\left(\mathbf{q}_2\right)\pi^{<}_{kl}\left(\mathbf{q}_3\right)$$

$$=\frac{K}{8}\left[(N-1)\int_{\Lambda/b}^{\Lambda}\frac{d^d q'}{(2\pi)^d}\frac{1}{Kq'^2}\right]\left(\int_0^{\Lambda/b}\frac{d^d q}{(2\pi)^d}q^2\sum_{j,k}\pi^{<}_{jk}\left(\mathbf{q}\right)\pi^{<}_{kj}\left(-\mathbf{q}\right)\right).$$

Adding up the two contributions results in an effective coupling

$$\frac{\tilde{K}}{4} = \frac{K}{4} + \frac{K}{8}N\int_{\Lambda/b}^{\Lambda}\frac{d^d q}{(2\pi)^d}\frac{1}{Kq^2},\qquad i.e.\qquad \tilde{K} = K + \frac{N}{2}I_d\left(b\right).$$

(h) Using the result from part (f), show that after matrix rescaling, the RG equation for K' is given by:

$$K' = b^{d-2}\left[K - \frac{N-2}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{q^2}\right]\quad .$$

• After coarse-graining, renormalizing the fields, and rescaling,

$$K' = b^{d-2}\zeta^2\tilde{K} = b^{d-2}\left[1 - \frac{N-1}{K}I_d\left(b\right)\right]K\left[1 + \frac{N}{2K}I_d\left(b\right)\right]$$

$$= b^{d-2}\left[K - \frac{N-2}{2}I_d\left(b\right) + \mathcal{O}\left(1/K\right)\right],$$

$i.e.$, to lowest non-trivial order,

$$K' = b^{d-2}\left[K - \frac{N-2}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^d q}{(2\pi)^d}\frac{1}{q^2}\right].$$

(i) Obtain the *differential* RG equation for $T = 1/K$, by considering $b = 1 + \delta\ell$. Sketch the flows for $d < 2$ and $d = 2$. For $d = 2 + \epsilon$, compute T_c and the critical exponent ν.

• Differential recursion relations are obtained for infinitesimal $b = 1 + \delta\ell$, as

$$K' = K + \frac{dK}{d\ell}\delta\ell = [1 + (d-2)\,\delta\ell]\left[K - \frac{N-2}{2}K_d\Lambda^{d-2}\delta\ell\right],$$

leading to

$$\frac{dK}{d\ell} = (d-2)\,K - \frac{N-2}{2}K_d\Lambda^{d-2}.$$

To obtain the corresponding equation for $T = 1/K$, we divide the above relation by $-K^2$, to get

$$\frac{dT}{d\ell} = (2-d)\,T + \frac{N-2}{2}K_d\Lambda^{d-2}T^2.$$

For $d < 2$, we have the two usual trivial fixed points: 0 (unstable) and ∞ (stable). The system is mapped unto higher temperatures by coarse-graining. The same applies for the case $d = 2$ and $N > 2$.

For $d > 2$, both 0 and ∞ are stable, and a non-trivial unstable fixed point appears at a finite temperature given by $dT/d\ell = 0$, $i.e.$

$$T^* = \frac{2\,(d-2)}{(N-2)\,K_d\Lambda^{d-2}} = \frac{4\pi\epsilon}{N-2} + \mathcal{O}\left(\epsilon^2\right).$$

In the vicinity of the fixed point, the flows are described by

$$\begin{aligned}\delta T' &= \left[1 + \frac{d}{dT}\left(\frac{dT}{d\ell}\right)\Big|_{T^*}\delta\ell\right]\delta T = \left\{1 + \left[(2-d) + (N-2)\,K_d\Lambda^{d-2}T^*\right]\delta\ell\right\}\delta T\\ &= (1 + \epsilon\delta\ell)\,\delta T.\end{aligned}$$

Thus, from

$$\delta T' = b^{y_T}\delta T = (1 + y_T\delta\ell)\,\delta T,$$

we get $y_T = \epsilon$, and

$$\nu = \frac{1}{\epsilon}.$$

(j) Consider a small symmetry breaking term $-h\int d^d\mathbf{x}\ \mathrm{tr}(M)$, added to the Hamiltonian. Find the renormalization of h, and identify the corresponding exponent y_h.

• As usual, h renormalizes according to

$$h' = b^d \zeta h = (1 + d\delta\ell)\left(1 - \frac{N-1}{2K} K_d \Lambda^{d-2} \delta\ell\right) h$$
$$= \left[1 + \left(d - \frac{N-1}{2K} K_d \Lambda^{d-2}\right)\delta\ell + \mathcal{O}\left(\delta\ell^2\right)\right] h.$$

From $h' = b^{y_h} h = (1 + y_h \delta\ell)\, h$, we obtain

$$y_h = d - \frac{N-1}{2K^*} K_d \Lambda^{d-2} = d - \frac{N-1}{N-2}(d-2) = 2 - \frac{\epsilon}{N-2} + \mathcal{O}\left(\epsilon^2\right).$$

Combining RG and symmetry arguments, it can be shown that the 3×3 matrix model is perturbatively equivalent to the $N = 4$ vector model at all orders. This would suggest that stacked triangular antiferromagnets provide a realization of the $\mathcal{O}(4)$ universality class; see P. Azaria, B. Delamotte, and T. Jolicoeur, J. Appl. Phys. **69**, 6170 (1991). However, non-perturbative (topological aspects) appear to remove this equivalence as discussed in S.V. Isakov, T. Senthil, Y.B. Kim, Phys. Rev. B **72**, 174417 (2005).

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11. *The roughening transition:* Consider a continuum interface model which in $d = 3$ is described by the Hamiltonian

$$\beta\mathcal{H}_0 = -\frac{K}{2}\int d^2\mathbf{x}\,(\nabla h)^2 \quad ,$$

where $h(\mathbf{x})$ is the interface height at location $\mathbf{x}$. For a crystalline facet, the allowed values of h are multiples of the lattice spacing. In the continuum, this tendency for integer h can be mimicked by adding a term

$$-\beta U = y_0 \int d^2\mathbf{x}\, \cos(2\pi h),$$

to the Hamiltonian. Treat $-\beta U$ as a perturbation, and proceed to construct a renormalization group as follows:

(a) Show that

$$\left\langle \exp\left[i \sum_\alpha q_\alpha h(\mathbf{x}_\alpha)\right]\right\rangle_0 = \exp\left[\frac{1}{K}\sum_{\alpha<\beta} q_\alpha q_\beta C(\mathbf{x}_\alpha - \mathbf{x}_\beta)\right]$$

for $\sum_\alpha q_\alpha = 0$, and zero otherwise. ($C(\mathbf{x}) = \ln|\mathbf{x}|/2\pi$ is the Coulomb interaction in two dimensions.)

• The translational invariance of the Hamiltonian constrains $\left\langle \exp\left[i\sum_\alpha q_\alpha h\left(\mathbf{x}_\alpha\right)\right]\right\rangle_0$ to vanish unless $\sum_\alpha q_\alpha = 0$, as implied by the following relation

$$\exp\left(i\delta\sum_\alpha q_\alpha\right)\left\langle\exp\left[i\sum_\alpha q_\alpha h\left(\mathbf{x}_\alpha\right)\right]\right\rangle_0 = \left\langle\exp\left\{i\sum_\alpha q_\alpha\left[h\left(\mathbf{x}_\alpha\right)+\delta\right]\right\}\right\rangle_0$$
$$= \left\langle\exp\left[i\sum_\alpha q_\alpha h\left(\mathbf{x}_\alpha\right)\right]\right\rangle_0 .$$

The last equality follows from the symmetry $\mathcal{H}\left[h\left(\mathbf{x}\right)+\delta\right] = \mathcal{H}\left[h\left(\mathbf{x}\right)\right]$. Using general properties of Gaussian averages, we can set

$$\left\langle\exp\left[i\sum_\alpha q_\alpha h\left(\mathbf{x}_\alpha\right)\right]\right\rangle_0 = \exp\left[-\frac{1}{2}\sum_{\alpha\beta} q_\alpha q_\beta\left\langle h\left(\mathbf{x}_\alpha\right)h\left(\mathbf{x}_\beta\right)\right\rangle_0\right]$$
$$= \exp\left[\frac{1}{4}\sum_{\alpha\beta} q_\alpha q_\beta\left\langle\left(h\left(\mathbf{x}_\alpha\right)-h\left(\mathbf{x}_\beta\right)\right)^2\right\rangle_0\right].$$

Note that the quantity $\left\langle h\left(\mathbf{x}_\alpha\right)h\left(\mathbf{x}_\beta\right)\right\rangle_0$ is ambiguous because of the symmetry $h(\mathbf{x}) \to h(\mathbf{x})+\delta$. When $\sum_\alpha q_\alpha = 0$, we can replace this quantity in the above sum with the height difference $\left\langle\left(h\left(\mathbf{x}_\alpha\right)-h\left(\mathbf{x}_\beta\right)\right)^2\right\rangle_0$ which is independent of this symmetry. (The ambiguity, or symmetry, results from the kernel of the quadratic form having a zero eigenvalue, which means that inverting it requires care.) We can now proceed as usual, and

$$\left\langle\exp\left[i\sum_\alpha q_\alpha h\left(\mathbf{x}_\alpha\right)\right]\right\rangle_0 = \exp\left[\sum_{\alpha,\beta}\frac{q_\alpha q_\beta}{4}\int\frac{d^2q}{\left(2\pi\right)^2}\frac{\left(e^{i\mathbf{q}\cdot\mathbf{x}_\alpha}-e^{i\mathbf{q}\cdot\mathbf{x}_\beta}\right)\left(e^{-i\mathbf{q}\cdot\mathbf{x}_\alpha}-e^{-i\mathbf{q}\cdot\mathbf{x}_\beta}\right)}{Kq^2}\right]$$
$$= \exp\left[\sum_{\alpha<\beta} q_\alpha q_\beta\int\frac{d^2q}{\left(2\pi\right)^2}\frac{1-\cos\left(\mathbf{q}\cdot\left(\mathbf{x}_\alpha-\mathbf{x}_\beta\right)\right)}{Kq^2}\right]$$
$$= \exp\left[\frac{1}{K}\sum_{\alpha<\beta} q_\alpha q_\beta C\left(\mathbf{x}_\alpha-\mathbf{x}_\beta\right)\right],$$

where

$$C\left(\mathbf{x}\right) = \int\frac{d^2q}{\left(2\pi\right)^2}\frac{1-\cos\left(\mathbf{q}\cdot\mathbf{x}\right)}{q^2} = \frac{1}{2\pi}\ln\frac{|\mathbf{x}|}{a},$$

is the Coulomb interaction in two dimensions, with a short distance cutoff a.

(b) Prove that

$$\left\langle|h(\mathbf{x})-h(\mathbf{y})|^2\right\rangle = -\left.\frac{d^2}{dk^2}G_k(\mathbf{x}-\mathbf{y})\right|_{k=0},$$

where $G_k(\mathbf{x}-\mathbf{y}) = \left\langle \exp\left[ik\big(h(\mathbf{x})-h(\mathbf{y})\big)\right]\right\rangle$.

• From the definition of $G_k(\mathbf{x}-\mathbf{y})$,

$$\frac{d^2}{dk^2}G_k(\mathbf{x}-\mathbf{y}) = -\left\langle \left[h(\mathbf{x})-h(\mathbf{y})\right]^2 \exp\left[ik\left(h(\mathbf{x})-h(\mathbf{y})\right)\right]\right\rangle.$$

Setting k to zero results in the identity

$$\left\langle \left[h(\mathbf{x})-h(\mathbf{y})\right]^2\right\rangle = -\left.\frac{d^2}{dk^2}G_k(\mathbf{x}-\mathbf{y})\right|_{k=0}.$$

(c) Use the results in (a) to calculate $G_k(\mathbf{x}-\mathbf{y})$ in perturbation theory to order of y_0^2. (Hint: Set $\cos(2\pi h) = \left(e^{2i\pi h}+e^{-2i\pi h}\right)/2$. The first order terms vanish according to the result in (a), while the second order contribution is identical in structure to that of the Coulomb gas described in this chapter.)

• Following the hint, we write the perturbation as

$$-U = y_0\int d^2x \cos(2\pi h) = \frac{y_0}{2}\int d^2x\left[e^{2i\pi h}+e^{-2i\pi h}\right].$$

The perturbation expansion for $G_k(\mathbf{x}-\mathbf{y}) = \left\langle \exp\left[ik\left(h(\mathbf{x})-h(\mathbf{y})\right)\right]\right\rangle \equiv \left\langle \mathcal{G}_k(\mathbf{x}-\mathbf{y})\right\rangle$ is calculated as

$$\begin{aligned}\left\langle \mathcal{G}_k\right\rangle =& \left\langle \mathcal{G}_k\right\rangle_0 - \left(\left\langle \mathcal{G}_k U\right\rangle_0 - \left\langle \mathcal{G}_k\right\rangle_0\left\langle U\right\rangle_0\right)\\ &+\frac{1}{2}\left(\left\langle \mathcal{G}_k U^2\right\rangle_0 - 2\left\langle \mathcal{G}_k U\right\rangle_0\left\langle U\right\rangle_0 + 2\left\langle \mathcal{G}_k\right\rangle_0\left\langle U\right\rangle_0^2 - \left\langle \mathcal{G}_k\right\rangle_0\left\langle U^2\right\rangle_0\right) + \mathcal{O}\left(U^3\right).\end{aligned}$$

From part (a),

$$\left\langle U\right\rangle_0 = \left\langle \mathcal{G}_k U\right\rangle_0 = 0,$$

and

$$\left\langle \mathcal{G}_k\right\rangle_0 = \exp\left[-\frac{k^2}{K}C(\mathbf{x}-\mathbf{y})\right] = \left(\frac{|\mathbf{x}-\mathbf{y}|}{a}\right)^{-\frac{k^2}{2\pi K}}.$$

Furthermore,

$$\begin{aligned}\left\langle U^2\right\rangle_0 &= \frac{y_0^2}{2}\int d^2\mathbf{x}' d^2\mathbf{x}''\left\langle \exp\left[2i\pi\left(h(\mathbf{x}')-h(\mathbf{x}'')\right)\right]\right\rangle_0\\ &= \frac{y_0^2}{2}\int d^2\mathbf{x}' d^2\mathbf{x}''\left\langle \mathcal{G}_{2\pi}(\mathbf{x}'-\mathbf{x}'')\right\rangle_0 = \frac{y_0^2}{2}\int d^2\mathbf{x}' d^2\mathbf{x}''\exp\left[-\frac{(2\pi)^2}{K}C(\mathbf{x}'-\mathbf{x}'')\right],\end{aligned}$$

and similarly,

$$\begin{aligned}&\left\langle \exp\left[ik\left(h(\mathbf{x})-h(\mathbf{y})\right)\right]U^2\right\rangle_0 =\\ &= \frac{y_0^2}{2}\int d^2\mathbf{x}' d^2\mathbf{x}''\exp\left\{-\frac{k^2}{K}C(\mathbf{x}-\mathbf{y}) - \frac{(2\pi)^2}{K}C(\mathbf{x}'-\mathbf{x}'')\right.\\ &\left.+\frac{2\pi k}{K}\left[C(\mathbf{x}-\mathbf{x}')+C(\mathbf{y}-\mathbf{x}'')\right] - \frac{2\pi k}{K}\left[C(\mathbf{x}-\mathbf{x}'')+C(\mathbf{y}-\mathbf{x}')\right]\right\}.\end{aligned}$$

Thus, the second order part of $G_k(\mathbf{x}-\mathbf{y})$ is

$$\frac{y_0^2}{4}\exp\left[-\frac{k^2}{K}C(\mathbf{x}-\mathbf{y})\right]\int d^2\mathbf{x}'d^2\mathbf{x}''\exp\left[-\frac{4\pi^2}{K}C(\mathbf{x}'-\mathbf{x}'')\right]\cdot$$
$$\cdot\left\{\exp\left[\frac{2\pi k}{K}\left(C(\mathbf{x}-\mathbf{x}')+C(\mathbf{y}-\mathbf{x}'')-C(\mathbf{x}-\mathbf{x}'')-C(\mathbf{y}-\mathbf{x}')\right)\right]-1\right\},$$

and

$$G_k(\mathbf{x}-\mathbf{y})=e^{-\frac{k^2}{K}C(\mathbf{x}-\mathbf{y})}\left\{1+\frac{y_0^2}{4}\int d^2\mathbf{x}'d^2\mathbf{x}''e^{-\frac{4\pi^2}{K}C(\mathbf{x}'-\mathbf{x}'')}\left(e^{\frac{2\pi k}{K}\mathcal{D}}-1\right)+\mathcal{O}\left(y_0^4\right)\right\},$$

where

$$\mathcal{D}=C(\mathbf{x}-\mathbf{x}')+C(\mathbf{y}-\mathbf{x}'')-C(\mathbf{x}-\mathbf{x}'')-C(\mathbf{y}-\mathbf{x}').$$

(d) Write the perturbation result in terms of an effective interaction K, and show that perturbation theory fails for K larger than a critical K_c.

• The above expression for $G_k(\mathbf{x}-\mathbf{y})$ is very similar to that of obtained in dealing with the renormalization of the Coulomb gas of vortices in the XY model. Following the steps in the lecture notes, without further calculations, we find

$$\begin{aligned}G_k(\mathbf{x}-\mathbf{y})&=e^{-\frac{k^2}{K}C(\mathbf{x}-\mathbf{y})}\left\{1+\frac{y_0^2}{4}\times\frac{1}{2}\left(\frac{2\pi k}{K}\right)^2\times C(\mathbf{x}-\mathbf{y})\times 2\pi\int drr^3e^{-\frac{2\pi\ln(r/a)}{K}}\right\}\\&=e^{-\frac{k^2}{K}C(\mathbf{x}-\mathbf{y})}\left\{1+\frac{\pi^3k^2}{K^2}y_0^2C(\mathbf{x}-\mathbf{y})\int drr^3e^{-\frac{2\pi\ln(r/a)}{K}}\right\}.\end{aligned}$$

The second order term can be exponentiated to contribute to an effective coupling constant K_{eff}, according to

$$\frac{1}{K_{\text{eff}}}=\frac{1}{K}-\frac{\pi^3}{K^2}a^{2\pi/K}y_0^2\int_a^\infty drr^{3-2\pi/K}.$$

Clearly, the perturbation theory is inconsistent if the above integral diverges, *i.e.* if

$$K>\frac{\pi}{2}\equiv K_c.$$

(e) Recast the perturbation result in part (d) into renormalization group equations for K and y_0, by changing the "lattice spacing" from a to ae^ℓ.

• After dividing the integral into two parts, from a to ab and from ab to ∞, respectively, and rescaling the variable of integration in the second part, in order to retrieve the usual limits of integration, we have

$$\frac{1}{K_{\text{eff}}}=\frac{1}{K}-\frac{\pi^3}{K^2}a^{2\pi/K}y_0^2\int_a^{ab}drr^{3-2\pi/K}-\frac{\pi^3}{K^2}a^{2\pi/K}\times y_0^2b^{4-2\pi/K}\times\int_a^\infty drr^{3-2\pi/K}.$$

(To order y_0^2, we can indifferently write K or K' (defined below) in the last term.) In other words, the coarse-grained system is described by an interaction identical in form, but parameterized by the renormalized quantities

$$\frac{1}{K'} = \frac{1}{K} - \frac{\pi^3}{K^2} a^{2\pi/K} y_0^2 \int_a^{ab} dr r^{3-2\pi/K},$$

and

$$y_0'^{\,2} = b^{4-2\pi/K} y_0^2.$$

With $b = e^\ell \approx 1 + \ell$, these RG relations are written as the following differential equations, which describe the renormalization group flows

$$\begin{cases} \dfrac{dK}{d\ell} = \pi^3 a^4 y_0^2 + \mathcal{O}\left(y_0^4\right) \\ \dfrac{dy_0}{d\ell} = \left(2 - \dfrac{\pi}{K}\right) y_0 + \mathcal{O}\left(y_0^3\right) \end{cases}.$$

(f) Using the recursion relations, discuss the phase diagram and phases of this model.

• These RG equations are similar to those of the XY model, with K (here) playing the role of T in the Coulomb gas. For non-vanishing y_0, K is relevant, and thus flows to larger and larger values (outside of the perturbative domain) if y_0 is also relevant ($K > \pi/2$), suggesting a smooth phase at low temperatures ($T \sim K^{-1}$). At small values of K, y_0 is irrelevant, and the flows terminate on a fixed line with $y_0 = 0$ and $K \leq \pi/2$, corresponding to a rough phase at high temperatures.

(g) For large separations $|\mathbf{x} - \mathbf{y}|$, find the magnitude of the discontinuous jump in $\left\langle |h(\mathbf{x}) - h(\mathbf{y})|^2 \right\rangle$ at the transition.

• We want to calculate the long distance correlations in the vicinity of the transition. Equivalently, we can compute the coarse-grained correlations. If the system is prepared at $K = \pi/2^-$ and $y_0 \approx 0$, under coarse-graining, $K \to \pi/2^-$ and $y_0 \to 0$, resulting in

$$G_k\left(\mathbf{x} - \mathbf{y}\right) \to \left\langle \mathcal{G}_k \right\rangle_0 = \exp\left[-\frac{2k^2}{\pi} C\left(\mathbf{x} - \mathbf{y}\right)\right].$$

From part (b),

$$\left\langle \left[h\left(\mathbf{x}\right) - h\left(\mathbf{y}\right)\right]^2 \right\rangle = -\frac{d^2}{dk^2} G_k\left(\mathbf{x} - \mathbf{y}\right)\bigg|_{k=0} = \frac{4}{\pi} C\left(\mathbf{x} - \mathbf{y}\right) = \frac{2}{\pi^2} \ln\left|\mathbf{x} - \mathbf{y}\right|.$$

On the other hand, if the system is prepared at $K = \pi/2^+$, then $K \to \infty$ under the RG (assuming that the relevance of K holds also away from the perturbative regime), and

$$\left\langle \left[h\left(\mathbf{x}\right) - h\left(\mathbf{y}\right)\right]^2 \right\rangle \to 0.$$

Thus, the magnitude of the jump in $\left\langle [h(\mathbf{x}) - h(\mathbf{y})]^2 \right\rangle$ at the transition is

$$\frac{2}{\pi^2} \ln |\mathbf{x} - \mathbf{y}| .$$

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12. *Roughening and duality:* Consider a discretized version of the Hamiltonian in the previous problem, in which for each site i of a square lattice there is an integer valued height h_i. The Hamiltonian is

$$\beta\mathcal{H} = \frac{K}{2} \sum_{<i,j>} |h_i - h_j|^\infty \quad ,$$

where the "∞" power means that there is no energy cost for $\Delta h = 0$; an energy cost of $K/2$ for $\Delta h = \pm 1$; and $\Delta h = \pm 2$ or higher *are not allowed* for neighboring sites. (This is known as the restricted solid on solid (RSOS) model.)

(a) Construct the dual model either diagrammatically, or by following these steps:

(i) Change from the N site variables h_i, to the $2N$ bond variables $n_{ij} = h_i - h_j$. Show that the sum of n_{ij} around any plaquette is constrained to be zero.

(ii) Impose the constraints by using the identity $\int_0^{2\pi} d\theta e^{i\theta n}/2\pi = \delta_{n,0}$, for integer n.

(iii) After imposing the constraints, you can sum freely over the bond variables n_{ij} to obtain a dual interaction $\tilde{v}(\theta_i - \theta_j)$ between dual variables θ_i on neighboring plaquettes.

• (i) In terms of bond variables $n_{ij} = h_i - h_j$, the Hamiltonian is written as

$$-\beta\mathcal{H} = -\frac{K}{2} \sum_{\langle ij \rangle} |n_{ij}|^\infty .$$

Clearly,

$$\sum_{\text{any closed loop}} n_{ij} = h_{i_1} - h_{i_2} + h_{i_2} - h_{i_3} + \cdots + h_{i_{n-1}} - h_{i_n} = 0,$$

since $h_{i_1} = h_{i_n}$ for a closed path.

(ii) This constraint, applied to the N plaquettes, reduces the number of degrees of freedom from an apparent $2N$ (bonds), to the correct figure N, and the partition function becomes

$$Z = \sum_{\{n_{ij}\}} e^{-\beta\mathcal{H}} \prod_\alpha \delta_{\sum_{\langle ij \rangle} n_{ij}^\alpha, 0},$$

where the index α labels the N plaquettes, and n_{ij}^α is non-zero and equal to n_{ij} only if the bond $\langle ij \rangle$ belongs to plaquette α. Expressing the Kronecker delta in its exponential representation, we get

$$Z = \sum_{\{n_{ij}\}} e^{-\frac{K}{2} \sum_{\langle ij \rangle} |n_{ij}|^\infty} \prod_\alpha \left(\int_0^{2\pi} \frac{d\theta_\alpha}{2\pi} e^{i\theta_\alpha \sum_{\langle ij \rangle} n_{ij}^\alpha} \right).$$

(iii) As each bond belongs to two neighboring plaquettes, we can label the bonds by $\alpha\beta$ rather than ij, leading to

$$Z = \left(\prod_\gamma \int_0^{2\pi} \frac{d\theta_\gamma}{2\pi}\right) \sum_{\{n_{\alpha\beta}\}} \exp\left(\sum_{\langle\alpha\beta\rangle} \left\{-\frac{K}{2}\left|n_{\alpha\beta}\right|^\infty + i\left(\theta_\alpha - \theta_\beta\right) n_{\alpha\beta}\right\}\right)$$
$$= \left(\prod_\gamma \int_0^{2\pi} \frac{d\theta_\gamma}{2\pi}\right) \prod_{\langle\alpha\beta\rangle} \sum_{n_{\alpha\beta}} \exp\left(\left\{-\frac{K}{2}\left|n_{\alpha\beta}\right|^\infty + i\left(\theta_\alpha - \theta_\beta\right) n_{\alpha\beta}\right\}\right).$$

Note that if all plaquettes are traversed in the same sense, the variable $n_{\alpha\beta}$ occurs in opposite senses (with opposite signs) for the constraint variables θ_α and θ_β on neighboring plaquettes. We can now sum freely over the bond variables, and since

$$\sum_{n=0,+1,-1} \exp\left(-\frac{K}{2}|n| + i\left(\theta_\alpha - \theta_\beta\right) n\right) = 1 + 2e^{-\frac{K}{2}} \cos\left(\theta_\alpha - \theta_\beta\right),$$

we obtain

$$Z = \left(\prod_\gamma \int_0^{2\pi} \frac{d\theta_\gamma}{2\pi}\right) \exp\left(\sum_{\langle\alpha\beta\rangle} \ln\left[1 + 2e^{-\frac{K}{2}} \cos\left(\theta_\alpha - \theta_\beta\right)\right]\right).$$

(b) Show that for large K, the dual problem is just the XY model. Is this conclusion consistent with the renormalization group results of the previous problem? (Also note the connection with the loop model.)

• This is the loop gas model , and for K large,

$$\ln\left[1 + 2e^{-\frac{K}{2}} \cos\left(\theta_\alpha - \theta_\beta\right)\right] \approx 2e^{-\frac{K}{2}} \cos\left(\theta_\alpha - \theta_\beta\right),$$

and

$$Z = \left(\prod_\gamma \int_0^{2\pi} \frac{d\theta_\gamma}{2\pi}\right) e^{\sum_{\langle\alpha\beta\rangle} 2e^{-\frac{K}{2}} \cos(\theta_\alpha - \theta_\beta)}.$$

This is none other than the partition function for the XY model, if we identify

$$K_{\rm XY} = 4e^{-\frac{K}{2}},$$

consistent with the results of another problem, in which we found that the low temperature behavior in the roughening problem corresponds to the high temperature phase in the XY model, and vice versa.

(c) Does the one dimensional version of this Hamiltonian, i.e. a 2d interface with

$$-\beta\mathcal{H} = -\frac{K}{2}\sum_i \left|h_i - h_{i+1}\right|^\infty \quad ,$$

have a roughening transition?

• In one dimension, we can directly sum the partition function, as

$$
\begin{aligned}
Z &= \sum_{\{h_i\}} \exp\left(-\frac{K}{2}\sum_i |h_i - h_{i+1}|^\infty\right) = \sum_{\{n_i\}} \exp\left(-\frac{K}{2}\sum_i |n_i|^\infty\right) \\
&= \prod_i \sum_{n_i} \exp\left(-\frac{K}{2}|n_i|^\infty\right) = \prod_i \left(1 + 2e^{-K/2}\right) = \left(1 + 2e^{-K/2}\right)^N ,
\end{aligned}
$$

($n_i = h_i - h_{i+1}$). The expression thus obtained is an analytic function of K (for $0 < K < \infty$), in the $N \to \infty$ limit, and there is therefore no phase transition at a finite non-zero temperature.

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