Scaling, Perturbation, & Renormalization

1. *The nonlinear σ model* describes n component unit spins. As we shall demonstrate later, in $d = 2$ dimensions, the recursion relations for temperature T, and magnetic field h, are

$$\begin{cases} \frac{dT}{d\ell} = \frac{(n-2)}{2\pi} T^2 \quad , \\ \frac{dh}{d\ell} = 2h \quad . \end{cases}$$

Note that T is a marginal operator, its relevance determined by the second order term.

(a) How does the correlation length diverge as $T \to 0$?

(b) Write down the singular form of the free energy as $T, h \to 0$.

(c) How does the susceptibility χ, diverge as $T \to 0$ for $h = 0$?

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2. *Coupled scalars:* Consider the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 - hm + \frac{L}{2} (\nabla^2 \phi)^2 + v \nabla m . \nabla \phi \right],$$

coupling two one component fields m and ϕ.

(a) Write $\beta\mathcal{H}$ in terms of the Fourier transforms $m(\mathbf{q})$ and $\phi(\mathbf{q})$.

(b) Construct a renormalization group transformation as in the text, by rescaling distances such that $\mathbf{q}' = b\mathbf{q}$; and the fields such that $m'(\mathbf{q}') = \tilde{m}(\mathbf{q})/z$ and $\phi'(\mathbf{q}') = \tilde{\phi}(\mathbf{q})/y$. Do not evaluate the integrals that just contribute a constant additive term.

(c) There is a fixed point such that $K' = K$ and $L' = L$. Find y_t, y_h and y_v at this fixed point.

(d) The singular part of the free energy has a scaling from $f(t, h, v) = t^{2-\alpha} g(h/t^\Delta, v/t^\omega)$ for t, h, v close to zero. Find α, Δ, and ω.

(e) There is another fixed point such that $t' = t$ and $L' = L$. What are the relevant operators at this fixed point, and how do they scale?

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3. *Anisotropic criticality:* A number of materials, such as liquid crystals, are anisotropic and behave differently along distinct directions, which shall be denoted parallel and perpendicular, respectively. Let us assume that the d spatial dimensions are grouped into n parallel directions $\mathbf{x}_{\parallel}$, and $d-n$ perpendicular directions $\mathbf{x}_{\perp}$. Consider a one–component field $m(\mathbf{x}_{\parallel}, \mathbf{x}_{\perp})$ subject to a Landau–Ginzburg Hamiltonian, $\beta\mathcal{H} = \beta\mathcal{H}_0 + U$, with

$$\beta\mathcal{H}_0 = \int d^n\mathbf{x}_{\parallel} d^{d-n}\mathbf{x}_{\perp} \left[\frac{K}{2}(\nabla_{\parallel}m)^2 + \frac{L}{2}(\nabla_{\perp}^2 m)^2 + \frac{t}{2}m^2 - hm\right],$$

and

$$U = u\int d^n\mathbf{x}_{\parallel} d^{d-n}\mathbf{x}_{\perp}\ m^4 .$$

(Note that $\beta\mathcal{H}$ depends on the **first** *gradient in the $\mathbf{x}_{\parallel}$ directions, and on the* **second** *gradient in the $\mathbf{x}_{\perp}$ directions.)*

(a) Write $\beta\mathcal{H}_0$ in terms of the Fourier transforms $m(\mathbf{q}_{\parallel}, \mathbf{q}_{\perp})$.

(b) Construct a renormalization group transformation for $\beta\mathcal{H}_0$, by rescaling coordinates such that $\mathbf{q}'_{\parallel} = b\ \mathbf{q}_{\parallel}$ and $\mathbf{q}'_{\perp} = c\ \mathbf{q}_{\perp}$ and the field as $m'(\mathbf{q}') = m(\mathbf{q})/z$. *Note that parallel and perpendicular directions are scaled differently.* Write down the recursion relations for K, L, t, and h in terms of b, c, and z. (The exact shape of the Brillouin zone is immaterial at this stage, and you do not need to evaluate the integral that contributes an additive constant.)

(c) Choose $c(b)$ and $z(b)$ such that $K' = K$ and $L' = L$. At the resulting fixed point calculate the eigenvalues y_t and y_h for the rescalings of t and h.

(d) Write the relationship between the (singular parts of) free energies $f(t, h)$ and $f'(t', h')$ in the original and rescaled problems. Hence write the unperturbed free energy in the homogeneous form $f(t, h) = t^{2-\alpha}g_f(h/t^{\Delta})$, and identify the exponents α and Δ.

(e) How does the unperturbed zero–field susceptibility $\chi(t, h = 0)$, diverge as $t \to 0$?

In the remainder of this problem set $h = 0$, and treat U as a perturbation.

(f) In the unperturbed Hamiltonian calculate the expectation value $\langle m(q)m(q')\rangle_0$, and the corresponding susceptibility $\chi_0(q) = \langle |m_q|^2\rangle_0$, where q stands for $(\mathbf{q}_{\parallel}, \mathbf{q}_{\perp})$.

(g) Write the perturbation U, in terms of the normal modes $m(q)$.

(h) Using RG, or any other method, find the upper critical dimension d_u, for validity of the Gaussian exponents.

(i) Write down the expansion for $\langle m(q)m(q')\rangle$, to first order in U, and reduce the correction term to a product of two point expectation values.

(j) Write down the expression for $\chi(q)$, in first order perturbation theory, and identify the transition point t_c at order of u. (Do not evaluate any integrals explicitly.)

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4. *Long–range interactions* between spins can be described by adding a term

$$\int d^d\mathbf{x} \int d^d\mathbf{y} J(|\mathbf{x}-\mathbf{y}|)\vec{m}(\mathbf{x})\cdot\vec{m}(\mathbf{y}),$$

to the usual Landau–Ginzburg Hamiltonian.

(a) Show that for $J(r) \propto 1/r^{d+\sigma}$, the Hamiltonian can be written as

$$\begin{aligned}\beta\mathcal{H} = &\int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{t+K_2q^2+K_\sigma q^\sigma+\cdots}{2}\vec{m}(\mathbf{q})\cdot\vec{m}(-\mathbf{q}) \\ &+u\int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3}{(2\pi)^{3d}}\vec{m}(\mathbf{q}_1)\cdot\vec{m}(\mathbf{q}_2)\,\vec{m}(\mathbf{q}_3)\cdot\vec{m}(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) \quad .\end{aligned}$$

(b) For $u = 0$, construct the recursion relations for (t, K_2, K_σ) and show that K_σ is irrelevant for $\sigma > 2$. What is the fixed Hamiltonian in this case?

(c) For $\sigma < 2$ and $u = 0$, show that the spin rescaling factor must be chosen such that $K'_\sigma = K_\sigma$, in which case K_2 is irrelevant. What is the fixed Hamiltonian now?

(d) For $\sigma < 2$, calculate the generalized Gaussian exponents ν, η, and γ from the recursion relations. Show that u is irrelevant, and hence the Gaussian results are valid, for $d > 2\sigma$.

(e) For $\sigma < 2$, use a perturbation expansion in u to construct the recursion relations for (t, K_σ, u) as in the text.

(f) For $d < 2\sigma$, calculate the critical exponents ν and η to first order in $\epsilon = d - 2\sigma$.

[See M.E. Fisher, S.-K. Ma and B.G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).]

(g) What is the critical behavior if $J(r) \propto \exp(-r/a)$? Explain!

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