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PROFESSOR:

Last time, we looked at the case of a Bose gas, and we found that there was this condensation phenomena into the ground state. Let's go over it one more time.

So the idea was that when you have a Bose system, the occupation of the different one-particle states, which you can specify for a non-interacting system in the grand canonical ensemble, these entities are independent for each one of the one-particle states and have a form that is related to the chemical potential and the energy of these states through this form, $e^{-\beta(\epsilon_k - \mu)}$ in the one-particle state, and z standing for $e^{\beta\mu}$, chemical potential divided by temperature. OK?

Now, if we go and look at this in the limit where z is much less than 1, which is appropriate to high temperatures or when the case where the gas is mostly classical, you can expand this. You can see that in the lowest order, it is $z e^{-\beta\epsilon_k}$. So it looks like a standard Boltzmann weight that we would use in classical statistical mechanics.

But then there are higher order terms. And in particular there would be a correction that is $e^{-\beta\epsilon_k} \frac{\beta^2 \epsilon_k^2}{2}$ and so forth that would modify this result. But I think this is something that I mentioned. If you look at a form such as this at high temperatures, it's like a Boltzmann weight.

So the state that has this lowest energy, in this case if we say ϵ_k is $\frac{\hbar^2 k^2}{2m}$ corresponding to k equals to 0 for the lowest energy, has the highest mean occupation number of weight. OK? Now that is always the case. This is a function that, as ϵ_k becomes larger, no matter whether you expand it or not, becomes less and less.

The other limit that is of interest to us is how big can z be? We said that the largest

it can be is when it approaches 1. And when it is 1, this quantity, the occupation number, is $e^{-\beta \epsilon_k}$. Again, something that is largest as k becomes smaller, and you go to the lowest energy.

But there is the difficulty that the one energy that corresponds to k equals to 0, if I look at this formula, it is 1 divided by 0. It just doesn't make too much sense.

Now, of course, we said that typically we don't want to work in this variable. We want to figure out things as a function of the number of particles or the number of particles divided by volume density, which is a nice intensive quantity. OK?

So typically, what should we do? We said that in the grand canonical ensemble, N is the sum over k these expectation values. I should really put an average here, but we said that in the thermodynamic sense, fluctuations being of the order of square root of N , we are going to ignore. So this is what you would normally call the number that we have in the system.

And then you have to replace this sum in the limit where we have a large system with an integral. So we have the integral $d^3k / (2\pi)^3 V$. And then we have this N of k , which is $1/z \int d^3k e^{-\beta \epsilon_k}$.

Now if I go like I did in the line above to the limit where z is much less than 1, then the integral that I have to do is simply the integral of $z e^{-\beta \epsilon_k}$. We know what this is. This is simply, up to a factor of z , these are the integrals that have given me these factors of $1/\lambda^3$. Actually, there's the factor of V here, so it is V/λ^3 .

So normally, you would then solve for z , and you would find that z is the density times λ^3 . And this is a formula that we have seen a number of times, that the chemical potential is $kt \log(n \lambda^3)$. OK?

Fine. But now we are interested in the other limit. Now in the other limit, we have this difficulty that this quantity, if I look at it as a function of z , this is the quantity that we would call f -- it is related to $f^{3/2}$ of z . So this is going to be $1/\lambda^3 f^{3/2}$ plus of z .

It's a function that increases with z up to the limiting value that we are allowed to have, which is at z equals to 1. It's a function that starts linearly, as we have discussed, but then up here, it comes to a finite value of $\zeta^{3/2}$. OK? So basically, we know that when I evaluate this for finite z , I'm less than where I was here. So this is certainly less than 1 over $\lambda^3 \zeta^{3/2}$, the limiting value that we have over here.

So now I have a difficulty. This function over here, I can only make it so big. But I need to make it big enough so that I can put enough particles, the total number of particles that I have in the system.

So what is happening over here? What is happening is that when I make this density larger and larger, ultimately when I hit this point at z equals to 1, as we have discussed there is an uncertainty as to what happens to this expression when z is equal to 1, and you are at this state that corresponds to k equals to 0. Right?

I don't know how many things are over there. So I can have, in principle, the freedom to take advantage of the fact that I don't know what 1 over 0 is to call that the number that I need to make up the difference between here and there. OK?

And so, in principle, when z goes to 1, then this becomes 1 over λ^3 , or V over $\lambda^3 \zeta^{3/2}$, obtained by essentially doing the integration with z equals to 1, evaluating it at this possible limit. And then whatever is left, I will put in the k equals to 0 state. OK?

And you will say, well, what does that explicitly mean? Well, what it means is that I'm really evaluating things at the value of z that is extremely close to 1 so that in the thermodynamic limit where N goes to infinity I wouldn't know the difference. But it is really something that vanishes only in that thermodynamic limit of N going to infinity.

And we saw that if I kind of look at the occupation number of the k equals to 0 state, what it is is z inverse, which is $e^{-\beta\mu}$. And if I'm in the limit where z is very small, or $\beta\mu$ is-- sorry, z is very much close to 1 or $\beta\mu$ is

very small, then I can expand this. And this becomes approximately, when μ goes to 0, minus 1 over $\beta \mu$. OK?

And I want this to be the N_0 , what is left over when I subtract from the total number the amount that I can put in to the ground state. OK? And so you can see that this is a quantity that is extensive. It is proportional to the number of particles. And μ is the inverse of that, so it is infinitesimal. And if I choose that value of μ that is, essentially, pushing z infinitesimally close to 1, then I can ensure that I have a finite occupation over there. OK?

So what does it mean? If I could actually look at these occupation numbers, then I am at temperatures less than T_c of n . And we saw that basically we could get what the temperature is when this happens by equating this combination g over λ^3 , in principle, over-- I had said g equals to 1. I could repeat it with g not equal to 1. And n , which is g over $\lambda^3 zeta$ of $3/2$, rather than f $3/2$ plus, this defines for me this temperature T_c of n . OK?

Now, if I could look at the occupation numbers when I was at temperature less than T_c , what would I see? I would see that the occupation number as a function of k is exactly the form that we have over here with z equals to 1, 1 over e to the $\beta \epsilon$ of k minus 1. But then at k equals to 0, which I can indicate by some kind of delta function, I have everybody else, which is this N minus V over $\lambda^3 zeta$ of $3/2$. OK?

Actually, let's remember that our k and momentum are related to each other. So rather than asking what the density is as a function of this label k , I can maybe ask what the density is for things that have different momentum P .

And according to this formula, it is going to be βP^2 over $2m$, this kind of modified version of the Boltzmann weight, and a delta function at P equals to 0 there. The rest of the particles are up here. Now, when people saw the signature of Bose-Einstein condensation, it was precisely this.

So this is the figure that shows this from the experiment of Eric Cornell and group,

and you see two thermometers. They are cooling the temperature of this gas of atoms that is confined in a trap. The way that they are cooling it is that they are evaporating away the high energy particles. And then the rest of the particles are cooled down because they thermalize, and the highest energy is always removed.

But in the process, the density in the trap is also being reduced. And if the density goes down, then T_c of n will also go down. So what is plotted over here on the right thermometer is the T_c of n that you would calculate from that formula. And on the right thermometer is actually the temperature of the particles in the gas.

And you can see that right at some point, then the T_c of the particles in the gas goes below T_c of n . And what is being plotted over here is as follows. So you have the particles that are in the trap at some temperature. We remove the trap, and then the particles are going to escape. The particles that have zero momentum are going to stay where they are. The particles that have large momentum are going to go further away.

So let's say after one second you take a picture. And what you see over here from the picture is over here there are the particles that had most momentum and ran further away. While at the center, you see the particles that had small momentum.

You can see that always, even at high temperature, the zero momentum is the peak. But once you go below this condition of T_c of n , in addition to the normal peak that is over here, you get some additional peak that is appearing at P equals to 0, which is the condensation that you have into that one single state. OK?

The other thing that I started to do last time but let's complete and finish today is I wanted to show you the heat capacity of this system. OK. So how do we do that? So the first formula that we had, the one that I used over here if I were to divide by V , is that the density, in general, is given by g over λ^3 $f^{3/2}$ plus of z . OK?

And we said that if z goes to 1, then I have to interpret that appropriately that this is really just excited states, and there's going to be a separation into the ground state and the excited states. So there is really two portions of n when we are below T_c .

OK?

But then there is also a formula for pressure. Beta P was g over λ cubed $f^{5/2}$ of z . Again, these f 's we had defined-- f plus m minus 1 of z was 1 over m minus 1 factorial integral 0 to infinity dx x to the m minus 1 z inverse e to the x plus 1-- minus 1. OK?

Fine. But in order to calculate the heat capacity, I need to work with the energy. And we said that, quite generally, whether you have classical system, quantum gas, bosons, fermions, the energy and pressure are related simply by this formula, which is ultimately a consequence of the scaling of the energy being proportional to momentum squared that we have over there. OK?

So let's look at this formula. What do I have? For T that is less than T_c of n , my z is equal to 1. Pressure comes entirely from particles that are moving around. So although the density breaks into two components, in principle I guess I would say that the pressure also breaks into two components. But the component that is at k equals to 0 is not moving around. It is not giving you any contribution to the pressure.

So what do we? We have P is kt , taking the beta to the other side. I have g over λ cubed, and then this f plus evaluated at z equals to 1, which is this zeta function of $5/2$. OK? So fine.

So what do I have for energy? Energy is going to be at $3/2$ V times this formula, kd T g over λ cubed $\zeta^{5/2}$. And I noticed that λ , being h square root of 2 π mk T , scales like T to the minus $1/2$. So 1 over λ cubed scales like T to the $3/2$ plus this is something that's scaling like T to the $5/2$.

And this is simply proportional to the volume and temperature to the [INAUDIBLE] fifth power. It knows nothing about the density. Why? Because if you were at the same temperature and volume, put more particle in the system, all of those additional particles would go over here, would make no contribution to pressure or to energy. You wouldn't see them.

So once you have this, the heat capacity at constant volume, dE by dT , is simply $\frac{5}{2}kT$ of whatever energy is. And there is a reason that I write it in this fashion.

So what you would say is that if I were to plot the heat capacity at constant volume as a function of temperature, well, its natural units are of the order of kB . You can see that this T and this T I can cancel out. So I have something that has units of kB . It is extensive proportional to volume. I can really certainly write it in this fashion.

And what I have is that the curve behaves like T to the $3/2$. So this is proportional to T to the $3/2$. And presumably, this behavior continues all the way until the temperature at which this system ceases to be a Bose-Einstein condensate. So the low temperature form of the heat capacity's calculation is very simple.

Now, at very high temperatures, we know that the heat capacity is going to be $\frac{3}{2} N kT$, right? So ultimately, I know that if I go to very high temperatures, my heat capacity is going to be $\frac{3}{2} N$, if I divide it by $[? kB. ?]$ OK?

Now, there is a reason that I drew this line over here less than the peak here. Well, in order to calculate the heat capacity on this side, we'll have to do a little bit more work. You can no longer assume that z equals to 1. z is something that is less than 1 that is obtained by solving N equals to g over $\lambda^3 f^{3/2}$ plus of z . OK?

But I can still presumably use these formulae. I have that the energy is $\frac{3}{2} PV$. So I have $V kT$ times-- what is the thing that I have on the right-hand side? It is g over λ^3 , just as I have over here except that I have to put $f^{5/2}$ plus of z . OK?

And if I want to calculate the heat capacity, what I need to do is to take a derivative of the energy with respect to temperature. And then I notice, well, again, right like I had before, whatever appears here is proportional to T to the $5/2$. So when I take a derivative with respect to temperature, I essentially get $5/2$ this whole quantity divided by T .

So just like I had over here, I will write the result as $\frac{3}{2} V kT g$ over λ^3 . And then I have the $5/2$ 1 over T of $f^{5/2}$ plus of z . But you say that's the wrong way of doing things because in addition to this combination out front be explicitly

proportional to T to the $5/2$ power, this factor z here is implicitly a function of temperature because it is related to density through that formula N equals to g over $\lambda^3 f^{5/2}$ plus of z .

So you should not forget to take the derivative of what is over here with respect to T . OK? So what do I get when I take a derivative of this with respect to T ? Well, the argument is not T , it is z . So I will have to take a dz by dT , and then the derivative of $f^{5/2}$ with respect to z .

And we saw that one of the characteristics of these functions was that when we took the derivative of each one of them, we generated one with a lower argument. So the derivative would be $f^{3/2}$ plus of z . But in order to have this [? latter ?] property, I had to do a z d by dz , and so I essentially need to restore this 1 over z here.

So you can see that, really, the difference between what I had at low temperatures and high temperatures, at low temperatures this was just a number that I was using over here, $zeta$ of $5/2$, and I didn't have to bother about this other term.

Now, we can figure out what the behavior of this term is certainly at small z because we have the expansion of these functions at small z . And if you do sufficient work, you will find-- well, first of all, at the largest values of z -- sorry, at the smallest value, z going to 0 , you can confirm that the heat capacity in units of $N k_B$ is simply $3/2$. So you will come to this line over here, and then the first correction that you calculate you'll find is going to take you in that direction.

Actually, if you do the calculation for bosons and fermions at high temperatures, you can convince yourself that while bosons will go in this direction, fermions, the heat capacity, will start to go in the other direction. But OK, so you have this behavior.

And then what we haven't established and the reason that I'm still doing all of this algebra is that, OK, this curve is going up, where is it going to end up? Is it going to end up here, here, or right where this curve hits? OK? And what I will show you is that this has to end up right at this point.

And actually, in order to show that, all we need to do is to figure out what these terms do when I evaluate them at z equals to 1. And because these things are telling me what is happening as I am approaching to this point, the question that I ask is, what is the limiting behavior of this function as I get to z equals to 1?

But the limiting value of this one I know. It is precisely what I want over here. It is ζ of $5/2$. So I have to somehow show you that this next term is 0. Well, this next term, however, depends on dz by dT . How can I get that? Well, all of these things are done under the conditions where the volume is fixed, or the number of particles is fixed, if the density is fixed.

So z and T are related through the formula that says N over V , which is the density, is g over $\lambda^3 f^{3/2}$ plus of z . OK? So if I take the system and I change its temperature, while keeping the number of particles, volume, fixed so that the density is fixed, if I take d by dT on both sides, on the other side I'm assuming that I'm doing the calculations under conditions where the density does not change.

And then what do I have over here? I have to take the temperature derivative of this. There's an implicit derivative here since this is T to the $3/2$ proportionately. So when I take a derivative, I will get 3 over $2T$, and then I will have $f^{3/2}$ plus of z .

But then I have to take the derivative that is implicit in the dependence on z here. And that's going to be identical to this, except that when I take a derivative of $f^{3/2}$, I will generate $f^{1/2}$ of z . OK? So since the left-hand side is equal to 0, I can immediately solve for this, and I find that $T dz$ by dT 1 over z is minus $3/2 f^{3/2}$ plus of z divided by $f^{1/2}$ plus of z . OK?

So essentially, what I can do is I can multiply through by T . So this T will get rid of this T . I can bring a T , therefore, here. And this combination over here I can replace from what I have there with minus $3/2 f^{3/2}$ plus of z divided by $f^{1/2}$ half plus of z .

Why is any of this useful? Well, it immediately gives the answer that I need. Why? Because $f^{1/2}$ is the derivative of this function. And you can see that I drew it so that it has infinite derivative when z hits 1. And we had indeed said that this function ζ

of m , which is the limit of this when z goes to 1, only exists for m that is larger than 1, and it is divergent for m that is less than 1. So when z hits 1, this thing goes to infinity, and the fraction disappears.

And so basically, the additional correction that you have beyond this becoming zeta of $5/2$, which is what we have over here, vanishes. So I have shown that, indeed, whatever this curve is doing, it will hit that point exactly at T_c of n at the same height as the low temperature curve. And you can do a certain amount of work and algebra to show that it goes there with a finite slope. So that's the shape of the heat capacity curve for the Bose-Einstein condensate.

AUDIENCE: Professor?

PROFESSOR: Yes?

AUDIENCE: So it seems that $f^{1/2}$ goes to infinity, but it's [? on ?] the half, so it is half comes from the [INAUDIBLE] 3. So--

PROFESSOR: Exactly.

AUDIENCE: --you have higher dimensional system, then it seems that [INAUDIBLE] won't vanish there. [INAUDIBLE].

PROFESSOR: So if you have a higher dimensional system, indeed the situation would be different. And I guess the borderline dimension would need to be-- since this is a V over 2 minus 1--

AUDIENCE: [INAUDIBLE]

PROFESSOR: V over 2 minus 1 being 1, which is the borderline, will give you a dc of 4. So something else happens above 4. Yes?

AUDIENCE: So it seems that the nature of this phase transition, indeed it changes with the dimension.

PROFESSOR: Exactly, yes. The nature of this phase transition does depend on [INAUDIBLE].

Good. OK? So we will come back to this heat capacity curve a little bit later because I want to now change directions and not talk about Bose-Einstein condensation anymore, but about a phenomena that is much more miraculous in my view, and that is superfluidity.

So the first thing that is very interesting about helium is its phase diagram. So we are used to drawing phase diagrams as a function of pressure and temperature, which have a separation between a liquid and a gas that terminates at a critical point.

So we have seen something like this many times, where you have on the low pressure/high temperature side a gas. On the higher pressure/lower temperature side, you have a liquid. And they terminate, this line of coexistence between the two of them, at a critical point. And for the case of helium, the critical point occurs roughly at 5.2 degrees Kelvin and something like 2.6 atmospheres. OK?

So far, nothing different from, say, liquid water, for example, and steam, except that for all other substances, when we cool it at the low enough temperatures, we go past the triple point, and then we have a solid phase.

For the case of helium, it stays liquid all the way down to 0 temperature. If you put sufficiently high pressure on it, ultimately it will become a solid. But at normal pressures-- so this maximal pressure is something of the order of 26 atmosphere. For pressures that are less than 26 atmosphere, you can cool the liquid all the way down to 0 temperature, and it stays happily a liquid.

So let's think about why that is the case. Well, why do things typically become solids anyway? The reason is that if I think about the interaction that I have between two objects, two atoms and molecules, we have discussed this in connection with condensation before that what we have is that at very large separations there is a van der Waals' attraction. Typically, the dipole moments of these atoms would be fluctuating, and the fluctuations would somehow align them temporarily in the manner that the energy can be reduced.

At very short distances, the electronic clouds will overlap, and so you would have something such as this. And then, typically, you will have the function that joins them and look something like this. OK? And this is no different for the case of helium. You have here essentially a hard core appearing at the distance that is of the order of 2.6 angstroms. There is a minimum that is of the order of 3 angstroms.

Now, the depth of this potential is not very big for helium. It is only of the order of, say, 9 degrees Kelvin. And the reason is that the strength of this van der Waals' attraction is proportional to the polarizability that scales typically with the number of electrons that you have in your atom.

And so it is, in fact, proportional to your atomic number squared. And with the exception of hydrogen, this is going to be the smallest value that you are going to get for the van der Waals' attraction. And hydrogen does not count because it forms a molecule.

So helium does not form a molecule. You have these [? balls. ?] They have these attractions. Now, typically things become solid as you go to 0 temperature is because you can find the minimum energy in which you put everybody at the minimum energy separation. So you form a lattice. Everybody is around this separation, and they're all happy and that solid minimizes the energy.

So why doesn't helium do the same thing? Maybe it should just do it at a lower temperature because the scale of this is lower. Again, the reason it doesn't do that is because of quantum mechanics. Because you can say that, really, I cannot specify exactly where the particles are because that would violate quantum mechanics.

There has to be some uncertainty Δx . And associated with Δx there is going to be some kind of a typical momentum, which is $\hbar / \Delta x$. And there will be some contribution to the kinetic energy of particles due to this uncertainty in Δx that is of this form.

And if I make the localization more precise, I have to pay a higher cost in kinetic

energy. Well, presumably the scale of the energies that I can tolerate and still keep these things localized is of the order of this binding energy, which is of the order of, as we said, 9 or 10 degrees Kelvin. Let's say 10 degrees Kelvin. OK?

Now, the other property that distinguishes helium from krypton and, let's say, all the other noble gases is that its mass is very small. So because this mass is small, this uncertainty in energy becomes comparatively larger. OK?

And so if you put the mass of the helium here and you use this value of 10 degrees K converted to the appropriate energy units, you ask what kind of a Δx would be compatible with this kind of binding energy, you find that the Δx that you get is of the order of 0.5 angstroms. OK? So basically, you cannot keep this in the bottom of the potential if you have uncertainty that is of this order.

So the atoms of helium cannot form a nice lattice and stay in order because that registry would violate the quantum uncertainty principle. They are moving around. And so you have a system that has been [? melted ?] due to quantum fluctuations rather than thermal fluctuations. OK?

So that's all nice, except that something does happen when you cool helium. And actually, it is interesting that helium provides us not only with one liquid that persists all the way to 0 temperature, but two liquids because helium has two isotopes. There is helium-4, which is the abundant one, and there is helium-3, which is a much less available isotope.

But the number of protons being different in the nucleus of these means that helium-4 is a boson, while helium-3 is fermion. So we have the opportunity to observe both a Bose system that is liquid at low temperature and a Fermi liquid at [INAUDIBLE] temperatures.

Now, the case of the Fermi liquid is not really that different in principle from the Fermi liquid of the electrons in copper that we already discussed. There are some subtleties. In the case of the bosons, [? well, ?] we saw that for bosons there is potentially this kind of phase transition.

In reality, what happens is that helium has a transition between two forms of the liquid at a temperature that is of the order of 2.18 degrees Kelvin. So we can call this helium I and this helium II, OK, two different forms of the helium-4 liquid that you have.

So how was this seen experimentally? I kind of told you how in these experiments of the Bose-Einstein condensation, part of the way that they cooled the system was to remove the high-energy particles, and the remaining particles cooled down. So that's the common method that actually people use for cooling various things. It's called evaporative-- evaporative cooling.

And for the case of helium, it is actually reasonably simple to draw what it is. You have a bath that is full of helium that is reasonably nicely thermally isolated from the environment. So you want to cool it, remove the energy from it. And the thing that you do is you pump out the gas. OK?

So this is a coexistence because this part up here on top of the liquid helium is not vacuum. What happens is that there is evaporation of helium, leaving the liquid and going into the gas. And once they are in the gas, they are pumped out.

But every time you grab a helium from here to put it into the gas, you've essentially taken it from down here and moved it to infinity, so you lose this amount of binding energy. It's called latent heat. Essentially, this is the latent heat, and it is actually of the order of 7 degrees Kelvin because typically it has to be the depth of the typical positions of these particles. OK?

So for each atom that you pump out, you have reduced the total energy that you have in the liquid, and the liquid cools down. And what is the trajectory? You always actually remain on this coexistence line. The pressure is changing. The temperature is lowering. But you're always remaining on the coexistence of the gas and liquid line.

So you follow what is happening over here, and you find that as you are approaching here, suddenly this system starts to bubble up. It is just like having a

kettle that is boiling. Indeed, it is, again, in like the case of kettle, there are these bubbles that are carrying the steam that is going to go over here and being pulled out, and so it is bubbling.

And then suddenly, [INAUDIBLE] heat 2.18 degrees Kelvin, and the whole thing quiets down. And below that, it stays quiet. And so let's see if we can show that. We saw the phase diagram. So this is the liquid. Let's see if we can get the sound also.

[VIDEO PLAYBACK]

-[INAUDIBLE]. This rapid [INAUDIBLE] evaporates [INAUDIBLE] helium cools, until at 2 degrees above absolute 0, a dramatic transformation takes place. Suddenly, we see--

PROFESSOR: Everything quiets.

-[INAUDIBLE] stops and that the surface of the liquid helium is completely still. The temperature is actually being lowered even further now, but nothing is happening. Well, this is really one of the great phenomenon in 20th-century physics.

PROFESSOR: OK. I'll show you this movie at the next lecture. But this is just a segment of it that shows the important property of this fluid, helium II, which is the superfluid. So now you're above the temperature it was boiling, but it quieted down. You can see that the fluid started to go through the capillaries that were at the bottom of this.

So at the bottom of this glass there were fine capillaries. As long as you were up here, the capillaries were sufficient to keep the fluid above. Once you went below, the fluid went through as if there is no resistance. And that's the superfluid property.

-Superfluidity and superconductivity were baffling concepts for scien--

[END PLAYBACK]

PROFESSOR: OK, we'll go back to that later.

So let's list some of the properties of this superfluid state. OK. So what this last

experiment showed is that it appears to have zero viscosity.

So what's the way that you would calculate the viscosity of a fluid? And one way to sort of do that relies on like what we had over here, passing the fluid through some pore or capillary. In the case of this experiment, you have to really make this capillary extremely fine so that practically nothing will go through it. But this could be a quite general way of measuring viscosity of some fluid.

So you have some fluid. You can measure its viscosity by trying to pass it through a capillary by exerting pressure on one side and measuring, in essence, the difference between the two pressures that you have on the two sides as a measure of the force that you are exerting on this.

And once you do that, then these pistons enhance the velocity-- sorry, the fluid, are moving through with some kind of velocity. And some measure of viscosity, after you divide by various things that have dimensions of length and pressure, et cetera, would be related to the velocity divided by ΔP -- is to achieve the same velocity, how much pressure difference you have to put.

Now, for this experiment of superfluid helium, you'll find that essentially you can get a finite velocity by typically infinitesimal pressure difference. So when you try to use this method to calculate viscosity, you find that, to whatever precision you can measure, essentially this value of η is not distinguishable from 0. OK? Yes?

AUDIENCE: The way you defined it, it would be infinity?

PROFESSOR: Then I probably defined it-- [? no? ?] ΔP goes to 0 for a finite V . Yes, so viscosity--

AUDIENCE: So what you're essentially writing is the Reynolds number, yeah?

PROFESSOR: Exactly.

AUDIENCE: OK.

PROFESSOR: All right? But there is another experiment that appears to indicate that there is a

finite viscosity, and that is a famous experiment due to Andronikashvili.

And basically, this is the other experiment. You imagine that you put your fluid inside a container. And then you insert in this container a set of plates, could be copper plates or something else, that are stacked on top of each other. So essentially, you have a stack of these plates that have a common axis, and there's a rod that is going through them.

And what you created here is a torsional oscillator in the sense that this rod that holds all of these things together has a particular preferred orientation. And if you start, let's say-- well, it may or may not have a particular orientation-- but if you start to rotate it, it will certainly have a moment of inertia that resists the rotation that is coming from all of the plates if it was in vacuum.

So if I could measure the moment of inertia of these objects, either through torsion or some other mechanism, if it is a torsional oscillator I could figure out what the frequency is. If I knew the [? spinning ?] constant that wants to bring the angle to some particular value, I would divide that by the moment of inertia.

The square root of that would give me the frequency. So the frequency is a measure of the moment of inertia. So if I had this thing in vacuum, I could figure out what the moment of inertia of all of the plates was.

Now, if I add the fluid to this, then as I start to rotate this, some of the fluid will certainly move with the plates. And so the moment of inertia that you would measure in the presence of the fluid is larger than it is in the absence of fluid. And that additional moment of inertia tells you about how much of the fluid is, in some sense, stuck to the plates, and that's another different measure of the viscosity.
OK?

So if we sort of look at this experiment, we would imagine that if I measure the moment of inertia as a function of temperature, something should happen when I hit T_c . Above T_c , I'm getting some value that is measuring, essentially, the moment of inertia of the plates plus the fluid that is moving around, [? we do. ?]

But what happens when you hit T_c ? If the superfluid doesn't care about things around it, you would imagine that this would drop down to the value that it has in vacuum. In reality, what it does is something like this.

So it does go to the value that you would have in vacuum but only at 0 temperature. So at any finite temperature we know T_c . There is indeed some fraction of the superfluid that is trapped by these plates. So what's happening? OK, so this is one [INAUDIBLE].

The other property-- maybe I will squeeze it somewhere here. There are a whole bunch of things that indicate that there is a coupling between temperature and mechanical properties, so-called thermo-mechanical couplings.

What you find in this experiment, where the fluid spontaneously goes, let's say, from this side to this side, so the direction is like this, is that when you measure the temperatures you will find that the temperature of this one goes up, and the temperature of this one goes down. So somehow the motion led to some temperature differences.

A dramatic one that, again, maybe we'll show in the movie next time is the fountain effect. What you do is you-- let's imagine that, again, you have a vat of helium, and inside of it you put a tube that is, let's say, shaped like this. And you have the fluid that is occupying everything here. Then inside this tube, you put some material that is a good absorber of heat, such as, let's say, a piece of carbon, and then you shine laser on it so that this thing becomes hot.

So in this case, the temperature goes up, and what you find happens is that the fluid will start to shoot up here like a fountain. It's called the fountain effect because of that. And essentially, whereas here pressure was converted to temperature, here temperature, the heating, gets converted to pressure.

So this is like-- I don't know if you've ever done this, where you put your hand under water, and if you squeeze your hand and the water shoots up. So it's kind of putting pressure here, causing a fountain that is shooting up like that. OK? So there is this

coupling between temperature and-- actually, this issue of boiling-not boiling is also a manifestation of a coupling between temperature and mechanical properties, which is whether the fluid is boiling or not boiling. OK?

So given everything that we have been talking about so far, it is tempting to imagine that there is a connection between all of this and the Bose-Einstein condensation.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yes?

AUDIENCE: Do superfluids have no vorticity or can they?

PROFESSOR: Yes, they can have vorticity, but it is of a special kind. It is quantized vorticity. And so what you can certainly do is you can take a vat of helium and set it into rotation, and then you will create vortex lines inside this superfluid.

AUDIENCE: OK. I was wondering if there was an analogy to superconductivity and magnetic fields.

PROFESSOR: The analog of the magnetic field is the rotation.

AUDIENCE: Right.

PROFESSOR: Yes.

AUDIENCE: OK.

PROFESSOR: So in that sense, yes. OK? OK. So similarities between BEC, Bose-Einstein Condensation, and superfluidity. I would say that perhaps the most important one, although I guess at the beginning of the story they didn't know this one, is that helium-3, which is a fermion, does not become superfluid around this same temperature of 2 degrees Kelvin, which is where helium-4 becomes superfluid.

What is special about this is, well, you would say that once you have mass and density, you can figure out what the Bose-Einstein temperature is. You had this formula that was $n \lambda^3$ is equal to $\zeta(3/2)$.

So that gives you some value of $k_B T_c$ being $\frac{h^2}{2\pi m}$ mass of helium, and then you have a zeta of $3/2$ and n somewhere. 3-- I guess this n -- oops, n -- yeah, it's correct-- n h^2 cubed $2\pi m k_B T_c$ to the $3/2$ being zeta of $3/2$. So $2\pi m k_B T_c$ is proportional to n zeta of $3/2$ to the $2/3$ power. OK.

So I had a picture here, which was the typical attraction potential. The typical separation of helium atoms is something like 3.6 angstroms. How do I know that is because the density I can figure out is typically 1 over the volume per particle, and this volume per particle is roughly 3.6 angstroms cubed. It's something like 46.2 angstrom cubed.

So I can put that value of n over here. I know what the mass of helium is. And I can figure out what this T_c is, and I find that T_c is 3.13 degrees K. You say, well, it's not exactly the 2π and something that we had.

But given that we completely ignored the interactions that we have between this, and this is the result that we have for [? point ?] particles of mass m with no interactions, this is actually very good. You would say, OK, this is roughly in the right range. And presumably if I were to include the effects of interactions, potentially I will get an interaction, temperature, that is roughly correct. So this is another reason.

The third reason is that you have automatically, when you are talking about the Bose-Einstein condensation, a relationship between pressure and temperature. So remember the formula that we had for the Bose-Einstein condensation, βP was something, where P is kT 1 over λ^3 zeta of $5/2$, right? So once you specify the temperature, you've also specified the pressure. They go hand in hand.

So then you can start to explain a lot of these phenomena, such as here you increase the temperature, pressure had to go up, and it was the increased pressure than did that. It kind of tells you something about what is going on in this experiment over here, where the temperature goes up where you have higher pressure.

It actually also explains why things stop bubbling when you go into the superfluid.

Because let's back out and think of why things are bubbling when you put your kettle on top of the oven. And the reason is that there are variations in temperature. There could be a local hotspot over here, and in this local hotspot, eventually there will be a bubble that forms, and the bubble goes away.

Why is it that it forms here is because heat gets spread out through diffusion. If I strike a match or even if I have a blow torch here, you won't feel the heat for some time. But if I do this, you hear my hand clapping much more rapidly. So pressure gets transmitted very efficiently through sound, whereas heat goes very slowly through diffusion. And that's why when you boil something, essentially you will have the diffusion of heat, local hotspots, bubbles.

But if temperature changes are connected to pressure changes, then for the superfluid, having a local hotspot is like doing this. It creates immediately a pressure change, and pressure changes are rapidly taken away. And that's why this could also explain what's going on over here.

So I think this number two was first noted by Fritz London. Maybe we'll talk about that a bit more next time around. Based on all of these similarities, Laszlo Tisza, who spent many years at MIT after being in Europe, proposed the two-fluid model.

So again, roughly inspired by what we saw for the case of the Bose-Einstein condensate, we saw that for the Bose-Einstein condensate, below T_c you had to make a separation for the number of things that were in the ground state, k equals to 0, and all of the others that were in the higher excited states. Pressure, heat capacity, everything came from the excited component, whereas the component that was at the k equals to 0 state was not making any contribution to these phenomena.

So what he proposed was that somehow, again, similarly when you are in the superfluid, you really have two components, two fluids that are in coexistence. There is a superfluid component, and there is a normal component.

And as you change temperature, these two components get converted to each

other. So just like the case of the Bose-Einstein condensate, it is not like you have so many of this fluid, so many of that fluid. These two things are always interchanging.

And then what is happening in this experiment, presumably, is that the superfluid component goes through, the normal component leaves behind. The superfluid component is the component that, in some sense, has less entropy, is at lower temperature. So when it goes here, it cools down this part. This part then goes up in temperature.

Again, there is no way that you can push all of the superfluid here and have normal here because these things get converted. So some amount of superfluid goes through here. Some amount gets converted from the normal back to the superfluid.

So there is this exchange that is constantly taking place in the system. And so on the basis of this model and having different velocities for the normal and superfluid component, Tisza could explain a number of these things. OK?

So I will mention some important differences at the end of this lecture and leave their resolution for next. So while those were similarities between BEC and superfluidity, there are very important differences. One important observation immediately is that if I try to compress superfluid helium or normal helium, it doesn't matter. It is just like trying to compress any other fluid such as water. It does not like to be squeezed.

So helium liquid is approximately, of course not entirely, incompressible, while BEC, we saw that the pressure is only a function of temperature. It doesn't know anything about the density. So you can change the density. All of the additional particles will go to go to the k equals to 0 state. It doesn't really care. So BEC is infinitely compressible.

And of course, this is immediately a manifestation of the interactions. In the case of the BEC, we didn't put any interactions among the particles. We can put as many of them as we like into the single state k equals to 0. Of course, real heliums, they

have the self-avoiding-- the interaction between them. They cannot go [? through ?] each other. So that's an important difference.

Number 2 is that detailed shapes/ T dependences of heat capacity and density of the superfluid part are different. OK. So I have up there what the heat capacity of a BEC should look like. Now, people have measured the heat capacity of helium as a function of temperature, and something does indeed happen at T_c . And what you find the shape is is something like this. It goes to 0, and then it does something like this.

And so while it doesn't look that different from what I have up there, except that when you look at the behavior down here, at the transition, over there it went to a finite value. Here it diverges as a log. So when you go to T_c , it actually goes to infinity. And because of the shape that this has, this is sometimes called the lambda transition.

Now, there's lots of issues about interesting things happening at the transition, and we won't be able to resolve that. But one thing that we should be able to figure out, based on what we know already, is the behavior as things go to 0 temperature. And I said that for the Bose-Einstein condensate, the temperature dependence is T to the $3/2$. What is important is that for the superfluid, it is actually proportional to T cubed.

And T cubed, if you go back in the course, is a signature of what was happening for phonons. So Landau looked at this and said there must be some kind of a phonon-like excitation going on, which is different from this case [INAUDIBLE] excitations that we have.

The other thing that we had was that if we look at the fraction that is in-- let's say the density that is in the k equals to 0 state as a function of temperature, what happens for the superfluid-- sorry, what happens for the Bose-Einstein condensate-- for the Bose-Einstein condensate, let's use the green-- essentially, the curve will come down to T_c linearly, and then it goes up, and it reaches its asymptotic value.

From this result, the fraction that is excited, the difference to 1 is proportional to $1/\lambda^3$. So this goes like $T^{-3/2}$. You can ask whether what was observed in the Andronikashvili experiment, eventually this measure also, the fraction that is in the superfluid component or excited states, does this curve match that curve?

But if I take this curve and put it over here, first of all, down here it is different. It goes like $T_c - T$ to the $2/3$ power rather than linear that you have over here. That, again, is less [? worry ?] than the fact that over here it goes to 0 proportional to T to the fourth. Again, something that needs explanation.

And finally, I will write down one statement, and then maybe we'll explain next time, is that BEC with the spectrum of excitations, which is $k^2/2m$, cannot be superfluid. There is an interesting reason for that that we will explain. And indeed, we'll see that having excitations that are [INAUDIBLE] like is compatible with superfluidity. This kind of excitation is not. So why is that, et cetera, we'll talk about next time.