Kinetic Theory

1. *Poisson Brackets:*

(a) Show that for observable $\mathcal{O}(\mathbf{p}(\mu), \mathbf{q}(\mu))$, $d\mathcal{O}/dt = \{\mathcal{O}, \mathcal{H}\}$, along the time trajectory of any micro state μ, where $\mathcal{H}$ is the Hamiltonian.

(b) If the ensemble average $\langle \{\mathcal{O}, \mathcal{H}\} \rangle = 0$ for any observable $\mathcal{O}(\mathbf{p}, \mathbf{q})$ in phase space, show that the ensemble density satisfies $\{\mathcal{H}, \rho\} = 0$.

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2. *Zeroth-order hydrodynamics:* The hydrodynamic equations resulting from the conservation of particle number, momentum, and energy in collisions are (in a uniform box):

$$\begin{cases} \partial_t n + \partial_\alpha \left(n u_\alpha \right) = 0 \\ \partial_t u_\alpha + u_\beta \partial_\beta u_\alpha = -\dfrac{1}{mn} \partial_\beta P_{\alpha\beta} \\ \partial_t \varepsilon + u_\alpha \partial_\alpha \varepsilon = -\dfrac{1}{n} \partial_\alpha h_\alpha - \dfrac{1}{n} P_{\alpha\beta} u_{\alpha\beta} \end{cases},$$

where n is the local density, $\vec{u} = \langle \vec{p}/m \rangle$, $u_{\alpha\beta} = \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha \right)/2$, and $\varepsilon = \left\langle mc^2/2 \right\rangle$, with $\vec{c} = \vec{p}/m - \vec{u}$.

(a) For the zeroth order density

$$f_1^0(\vec{p}, \vec{q}, t) = \frac{n(\vec{q}, t)}{\left(2\pi m k_B T(\vec{q}, t)\right)^{3/2}} \exp\left[-\frac{\left(\vec{p} - m\vec{u}(\vec{q}, t)\right)^2}{2mk_B T(\vec{q}, t)} \right],$$

calculate the pressure tensor $P^0_{\alpha\beta} = mn \left\langle c_\alpha c_\beta \right\rangle^0$, and the heat flux $h^0_\alpha = nm \left\langle c_\alpha c^2/2 \right\rangle^0$.

(b) Obtain the zeroth order hydrodynamic equations governing the evolution of $n(\vec{q}, t)$, $\vec{u}(\vec{q}, t)$, and $T(\vec{q}, t)$.

(c) Show that the above equations imply $D_t \ln\left(nT^{-3/2}\right) = 0$, where $D_t = \partial_t + u_\beta \partial_\beta$ is the material derivative along streamlines.

(d) Write down the expression for the function $\mathrm{H}^0(t) = \int d^3\vec{q} d^3\vec{p} f_1^0(\vec{p}, \vec{q}, t) \ln f_1^0(\vec{p}, \vec{q}, t)$, after performing the integrations over $\vec{p}$, in terms of $n(\vec{q}, t)$, $\vec{u}(\vec{q}, t)$, and $T(\vec{q}, t)$.

(e) Using the hydrodynamic equations in (b) calculate $d\mathrm{H}^0/dt$.

(f) Discuss the implications of the result in (e) for approach to equilibrium.

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3. *Viscosity:* Consider a classical gas between two plates separated by a distance w. One plate at $y = 0$ is stationary, while the other at $y = w$ moves with a constant velocity $v_x = u$. A zeroth order approximation to the one particle density is,

$$f_1^0(\vec{p}, \vec{q}) = \frac{n}{(2\pi m k_B T)^{3/2}} \exp\left[-\frac{1}{2mk_B T}\left((p_x - m\alpha y)^2 + p_y^2 + p_z^2\right)\right],$$

obtained from the *uniform* Maxwell–Boltzmann distribution by substituting the average value of the velocity at each point. ($\alpha = u/w$ is the velocity gradient.)

(a) The above approximation does not satisfy the Boltzmann equation as the collision term vanishes, while $df_1^0/dt \neq 0$. Find a better approximation, $f_1^1(\vec{p})$, by linearizing the Boltzmann equation, in the single collision time approximation, to

$$\mathcal{L}\left[f_1^1\right] \approx \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{q}}\right] f_1^0 \approx -\frac{f_1^1 - f_1^0}{\tau_\times},$$

where $\tau_\times$ is a characteristic mean time between collisions.

(b) Calculate the net transfer Π_{xy} of the x component of the momentum, of particles passing through a plane at y, per unit area and in unit time.

(c) Note that the answer to (b) is independent of y, indicating a uniform transverse force $F_x = -\Pi_{xy}$, exerted by the gas on each plate. Find the coefficient of viscosity, defined by $\eta = F_x/\alpha$.

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4. *Light and matter:* In this problem we use kinetic theory to explore the equilibrium between atoms and radiation.

(a) The atoms are assumed to be either in their ground state a_0, or in an excited state a_1, which has a higher energy ε. By considering the atoms as a collection of N fixed two-state systems of energy E (i.e. ignoring their coordinates and momenta), calculate the ratio n_1/n_0 of densities of atoms in the two states as a function of temperature T.

Consider photons γ of frequency $\omega = \varepsilon/\hbar$ and momentum $|\vec{p}| = \hbar\omega/c$, which can interact with the atoms through the following processes:

(i) *Spontaneous emission:* $a_1 \to a_0 + \gamma$.

(ii) *Adsorption:* $a_0 + \gamma \to a_1$.

(iii) *Stimulated emission:* $a_1 + \gamma \to a_0 + \gamma + \gamma$.

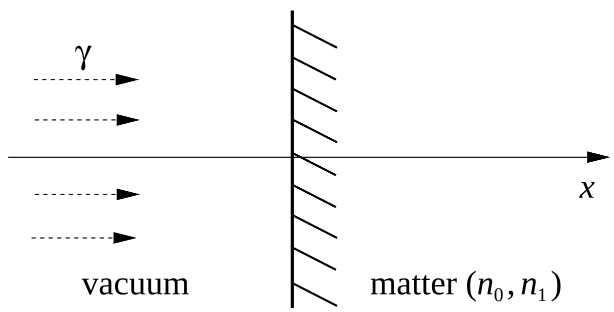
Assume that spontaneous emission occurs with a probability $\sigma_{\rm sp}$, and that adsorption and stimulated emission have corresponding constant (angle-independent) probabilities (cross-sections) of $\sigma_{\rm ad}$ and $\sigma_{\rm st}$, respectively.

(b) Write down the Boltzmann equation governing the density f of the photon gas, treating the atoms as fixed scatterers of densities n_0 and n_1.

(c) Find the equilibrium density $f_{\rm eq.}$ for the photons of the above frequency.

(d) According to Planck's law, the density of photons at a temperature T depends on their frequency ω as $f_{\rm eq.} = \left[\exp\left(\hbar\omega/k_B T\right) - 1\right]^{-1}/h^3$. What does this imply about the above cross sections?

(e) Consider a situation in which light shines along the x axis on a collection of atoms whose boundary coincides with the $x = 0$ plane, as illustrated in the figure.



Clearly, f will depend on x (and p_x), but will be independent of y and z. Adapt the Boltzmann equation you propose in part (b) to the case of a uniform incoming flux of photons with momentum $\vec{p} = \hbar\omega\hat{x}/c$. What is the *penetration length* across which the incoming flux decays?

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5. *Equilibrium density:* Consider a gas of N particles of mass m, in an external potential $U(\vec{q})$. Assume that the one body density $\rho_1(\vec{p}, \vec{q}, t)$, satisfies the Boltzmann equation. For a stationary solution, $\partial\rho_1/\partial t = 0$, it is *sufficient* from Liouville's theorem for ρ_1 to satisfy $\rho_1 \propto \exp\left[-\beta\left(p^2/2m + U(\vec{q})\right)\right]$. Prove that this condition is also *necessary* by using the H-theorem as follows.

(a) Find $\rho_1(\vec{p}, \vec{q})$ that minimizes $\mathrm{H} = N\int d^3\vec{p}d^3\vec{q}\rho_1(\vec{p}, \vec{q})\ln\rho_1(\vec{p}, \vec{q})$, subject to the constraint that the total energy $E = \langle\mathcal{H}\rangle$ is constant. (Hint: Use the method of Lagrange multipliers to impose the constraint.)

(b) For a mixture of two gases (particles of masses m_a and m_b) find the distributions $\rho_1^{(a)}$ and $\rho_1^{(b)}$ that minimize $\mathrm{H} = \mathrm{H}^{(a)} + \mathrm{H}^{(b)}$ subject to the constraint of constant total energy. Hence show that the kinetic energy per particle can serve as an empirical temperature.

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6. (Optional) *Electron emission:* When a metal is heated in vacuum, electrons are emitted from its surface. The metal is modeled as a classical gas of noninteracting electrons held in the solid by an abrupt potential well of depth ϕ (the work function) relative to the vacuum.

(a) What is the relationship between the initial and final velocities of an escaping electron?

(b) In thermal equilibrium at temperature T, what is the probability density function for the velocity of electrons?

(c) If the number density of electrons is n, calculate the current density of thermally emitted electrons.

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† Reviewing the problems and solutions provided on the course web-page for preparation for *Test 2* should help you with the above problems.