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**Probability**

**1. The book of records:** Consider a sequence of random numbers  $\{x_1, x_2, \dots, x_n, \dots\}$ ; the entry  $x_n$  is a *record* if it is larger than all numbers before it, i.e. if  $x_n > \{x_1, x_2, \dots, x_{n-1}\}$ . We can then define an associated sequence of indicators  $\{R_1, R_2, \dots, R_n, \dots\}$  in which  $R_n = 1$  if  $x_n$  is a record, and  $R_n = 0$  if it is not (clearly  $R_1 = 1$ ).

(a) Assume that each entry  $x_n$  is taken independently from the same probability distribution  $p(x)$ . [In other words,  $\{x_n\}$  are *IIDs* (independent identically distributed).] Show that, irrespective of the form of  $p(x)$ , there is a very simple expression for the probability  $P_n$  that the entry  $x_n$  is a record.

(b) The records are entered in the *Guinness Book of Records*. What is the average number  $\langle S_N \rangle$  of records after  $N$  attempts, and how does it grow for  $N \gg 1$ ? If the number of trials, e.g. the number of participants in a sporting event, doubles every year, how does the number of entries asymptotically grow with time.

(c) Prove that  $\langle R_n R_m \rangle_c = 0$  for  $m \neq n$ . (The record indicators  $\{R_n\}$  are in fact *independent* random variables, though not identical, which is a stronger statement than the vanishing of the covariance.)

(d) Compute all moments, and the first three cumulants of the total number of records  $S_N$  after  $N$  entries. Does the central limit theorem apply to  $S_N$ ?

(e) The first record, of course occurs for  $n_1 = 1$ . If the third record occurs at trial number  $n_3 = 9$ , what is the mean value  $\langle n_2 \rangle$  for the position of the second record? What is the mean value  $\langle n_4 \rangle$  for the position of the fourth record?

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**2. Dice:** A dice is loaded such that 6 occurs twice as often as 1.

(a) Calculate the unbiased probabilities for the 6 faces of the dice.

(b) What is the information content (in bits) of the above statement regarding the dice?

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**3. Random deposition:** A mirror is plated by evaporating a gold electrode in vacuum by passing an electric current. The gold atoms fly off in all directions, and a portion of them sticks to the glass (or to other gold atoms already on the glass plate). Assume that each

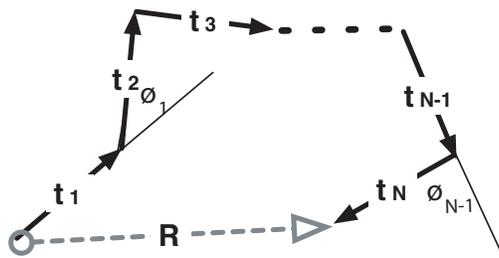
column of deposited atoms is independent of neighboring columns, and that the average deposition rate is  $d$  layers per second.

(a) What is the probability of  $m$  atoms deposited at a site after a time  $t$ ? What fraction of the glass is not covered by any gold atoms?

(b) What is the variance in the thickness?

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4. *Semi-flexible polymer in two dimensions* Configurations of a model polymer can be described by either a set of vectors  $\{\mathbf{t}_i\}$  of length  $a$  in two dimensions (for  $i = 1, \dots, N$ ), or alternatively by the angles  $\{\phi_i\}$  between successive vectors, as indicated in the figure below.



The polymer is at a temperature  $T$ , and subject to an energy

$$\mathcal{H} = -\kappa \sum_{i=1}^{N-1} \mathbf{t}_i \cdot \mathbf{t}_{i+1} = -\kappa a^2 \sum_{i=1}^{N-1} \cos \phi_i \quad ,$$

where  $\kappa$  is related to the bending rigidity, such the probability of any configuration is proportional to  $\exp(-\mathcal{H}/k_B T)$ .

(a) Show that  $\langle \mathbf{t}_m \cdot \mathbf{t}_n \rangle \propto \exp(-|n - m|/\xi)$ , and obtain an expression for the *persistence length*  $\ell_p = a\xi$ . (You can leave the answer as the ratio of simple integrals.)

(b) Consider the end-to-end distance  $\mathbf{R}$  as illustrated in the figure. Obtain an expression for  $\langle R^2 \rangle$  in the limit of  $N \gg 1$ .

(c) Find the probability  $p(\mathbf{R})$  in the limit of  $N \gg 1$ .

(d) If the ends of the polymer are pulled apart by a force  $\mathbf{F}$ , the probabilities for polymer configurations are modified by the Boltzmann weight  $\exp\left(\frac{\mathbf{F} \cdot \mathbf{R}}{k_B T}\right)$ . By expanding this weight, or otherwise, show that

$$\langle \mathbf{R} \rangle = K^{-1} \mathbf{F} + \mathcal{O}(F^3) \quad ,$$

and give an expression for the Hookian constant  $K$  in terms of quantities calculated before.

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**5. (Optional) *Simpson's paradox:*** In 1970's Berkeley was (wrongly) sued for bias against women applying for graduate admission due to a misuse of statistics. The following examples demonstrate how this can happen.

(a) Consider a university with only two departments E (easy admissions) and H (hard admissions); each admitting a higher proportion of female applications, according to the probabilities  $p_f^E > p_m^E > p_f^H > p_m^H$ . Show that the university as a whole can still end up admitting more men, if the men apply predominantly to E, while women apply predominantly to H.

(b) Use the same principle to construct a fake demonstration of violation of the second law as follows. An equal number of Carnot and non-Carnot engines are paired together; each pair operating with the same input/output sources (possibly different from other pairs). The same quantity of heat  $Q$  is separately distributed amongst the Carnot engines, and also amongst the non-Carnot engines. It is found that the net work produced by the non-Carnot engines is larger than that produced by the Carnot engines. How is this possible?

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**6. *Jarzynski equality:*** In equilibrium at a temperature  $T$ , the probability that a macroscopic system is in a microstate  $\mu$  is  $p(\mu) = \exp[-\beta\mathcal{H}(\mu)]/Z$ , where  $\mathcal{H}(\mu)$  is the energy of the microstate,  $\beta = 1/(k_B T)$ , and the normalization factor is related to the free energy by  $-\beta F = \ln Z$ . We now change the macroscopic state of the system by performing external work  $W$ , such that the new state is also in equilibrium at temperature  $T$ . For example, imagine that the volume of a gas is changed by moving a piston as  $L(t) = L_1 + (L_2 - L_1)t/\tau$ . Depending on the protocol (e.g. the speed  $u = (L_2 - L_1)/\tau$ ), the process may be close to or far from reversible. Nonetheless, the Jarzynski equality relates the probability distribution for the work  $W$  to the *equilibrium* change in free energy!

(a) Assume that the process by which work is performed is fully deterministic, in the sense that for a given protocol, any initial microstate  $\mu$  will evolve to a specific final microstate  $\mu'$ . The amount of work performed  $W(\mu)$  will vary with the initial microstate, and there is thus a probability distribution  $p_f(W)$  which can be related to the equilibrium  $p(\mu)$ . The energy of the final microstate, however, is precisely  $\mathcal{H}'(\mu') = \mathcal{H}(\mu) + W(\mu)$ . Time reversal

symmetry implies that if we now instantaneously reverse all the momenta, and proceed according to the reversed protocol, the time-reversed microstate  $\overline{\mu'}$  will deterministically evolve back to microstate  $\mu$ , and the work  $-W(\mu)$  is recovered. However, rather than doing so, we first allow the system to equilibrate into its new macrostate at temperature  $T$ , before reversing the protocol to recover the work. The recovered work  $-W$  will now be a function of the selected microstate, and distributed according to a different probability  $p_b(-W)$ , related to  $p'(\mu') = \exp[-\beta\mathcal{H}'(\mu')] / Z'$ . It is in general not possible to find  $p_f(W)$  or  $p_b(-W)$ . However, by noting that the probabilities of a pair of time-reversed microstates are exactly equal, show that their ratio is given by

$$\frac{p_f(W)}{p_b(-W)} = \exp[\beta(W + F - F')].$$

While you were guided to prove the above result with specific assumptions, it is in fact more generally valid, and known as the *work-fluctuation theorem*.

(b) Prove the *Jarzynski equality*

$$\Delta F \equiv F' - F = -k_B T \ln \langle e^{-\beta W} \rangle \equiv -k_B T \ln \left[ \int dW p_f(W) e^{-\beta W} \right].$$

This result can in principle be used to compute equilibrium free energy differences from an ensemble of non-equilibrium measurements of the work. For example, in *Liphardt, et al., Science* **296**, 1832 (2002), the work needed to stretch a single RNA molecule was calculated and related to the free energy change. However, the number of trials must be large enough to ensure that the averaged exponential, which is dominated by rare events, is accurately obtained.

(c) Use the Jarzynski equality to prove the familiar thermodynamic inequality

$$\langle W \rangle \geq \Delta F \quad .$$

(d) Consider a cycle in which a work  $W + \omega$  is performed in the first stage, and work  $-W$  is returned in the reversed process. According to the second law of thermodynamics, the net gain  $\omega$  must be negative. However, within statistical physics, it is always possible that this condition is violated. Use the above results to conclude that the probability of violating the second law decays with the degree of violation according to

$$P_{\text{violating second law}}(\omega) \leq e^{-\beta\omega}.$$

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† Two relevant articles: **(i)** On the Jarzynski relation, *G.E. Crooks and C. Jarzynski, Phys. Rev. E* **75**, 021116 (2007); **(ii)** Regarding records, *J. Krug, J. Stat. Mech.* (2007) P07001.

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