Chapter 3

Grand Unified Theory

3.1 SU(5) Unification

Gauge bosons:

$$\left(\begin{array}{c|c} SU(3) & \\ \hline & SU(2) \end{array}\right) \tag{3.1}$$

$U(1)$: (commuting with $SU(3) \times SU(2)$)

$$\begin{pmatrix} e^{2i\lambda} & & & & \\ & e^{2i\lambda} & & & \\ & & e^{2i\lambda} & & \\ & & & e^{-3i\lambda} & \\ & & & & e^{-3i\lambda} \end{pmatrix} \tag{3.2}$$

Or Lie algebra:

$$\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \tag{3.3}$$

One gets breaking $SU(5) \to SU(3) \times SU(2) \times U(1)$ in this pattern with an adjoint "Higgs" field obeying

$$< \Phi > \propto \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \tag{3.4}$$

What do the functions think of this?

$$\begin{pmatrix} u \\ d \end{pmatrix}_{L \quad \frac{1}{6}}, \begin{pmatrix} v \\ e \end{pmatrix}_{L \quad -\frac{1}{2}} \tag{3.5}$$

$$\begin{matrix} u_{R \quad \frac{2}{3}} \\ d_{R \quad -\frac{1}{3}} \\ e_{R \quad -1} \end{matrix} \quad \overset{change\ conjugation}{\longrightarrow} \quad \begin{matrix} (u_R^C)_{- \quad \frac{2}{3}} \\ (d_R^C)_{\frac{1}{3}} \\ (e_R^C)_1 \end{matrix} \tag{3.6}$$

all L – handed .
Multiplets? Clues:

$$15 = \underbrace{10}_{antisymmetric\ tensor} + \underbrace{5}_{vector} \tag{3.7}$$

$$\sum Y = 0 \tag{3.8}$$

Altogether?

$$6 \times \frac{1}{6} + 2 \times -\frac{1}{2} + 3 \times -\frac{2}{3} + 3 \times \frac{1}{3} + 1 \quad = \quad 0 \tag{3.9}$$

$$(d_R^C)_{\frac{1}{3}} + \begin{pmatrix} v \\ e \end{pmatrix}_{L \quad -\frac{1}{2}} \quad make \quad 5 \tag{3.10}$$

Actually (using $\sum_{\alpha\beta}$)

$$\begin{pmatrix} d_R^C \\ -e \\ v \end{pmatrix} = \bar{5} \tag{3.11}$$

Note:

$$\tilde{g} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} = g'Y \tag{3.12}$$

with

$$\tilde{g} = \frac{-g'}{6} \tag{3.13}$$

Can the residue be identified with 10? $\Psi^{\alpha i}, \alpha = 1,2,3, i = 4,5$, color: 3, $SU(2) : 2$.

$$Y = -\frac{1}{6}(2-3) = \frac{1}{6} \Rightarrow \begin{pmatrix} u \\ d \end{pmatrix}_L \tag{3.14}$$

$\Psi^{\alpha\beta}$, color: $\bar{3}$, $SU(2)$: singlet.

$$Y = -\frac{1}{6}(2+2) = -\frac{2}{3} \Rightarrow u_R^C \tag{3.15}$$

Ψ^{ij}, color: singlet, $SU(2) : 2$

$$Y = -\frac{1}{6}(-3-3) = 1 \Rightarrow e_R^C \tag{3.16}$$

It clicks.
Normalizing $\tilde{g}$:

$$SU(3)\ generators \quad : \quad g_{un.} \begin{pmatrix} -\frac{1}{2} & & & & \\ & -\frac{1}{2} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} etc. \tag{3.17}$$

$$SU(2)\ generators \quad : \quad g_{un.} \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \frac{1}{2} & \\ & & & & -\frac{1}{2} \end{pmatrix} etc. \tag{3.18}$$

$$tr\Gamma_a\Gamma_b \quad = \quad \frac{1}{2} f_{ab} \tag{3.19}$$

$$U(1)\ generators \quad : \quad \tilde{g} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

$$= g' \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{pmatrix} \tag{3.20}$$

$$g'^2(3 \cdot (\frac{1}{3})^2 + 2 \cdot (\frac{1}{2})^2) = \frac{1}{2} g^2_{un.} \tag{3.21}$$

$$g'^2(\frac{5}{6}) = \frac{1}{2} g^2_{un.} \tag{3.22}$$

$$g'^2 = \frac{3}{5} g^2_{un.} \tag{3.23}$$

So "naive" prediction:

$$g^2_S = g^2_\omega = \frac{5}{3} g^2_{un.} \tag{3.24}$$

$$\sin^2 \theta_\omega = \frac{g'^2}{g'^2 + g^2_\omega} = \frac{\frac{3}{5}}{1 + \frac{3}{5}} = \frac{3}{8} \tag{3.25}$$

Expt. ≈ 0.22.

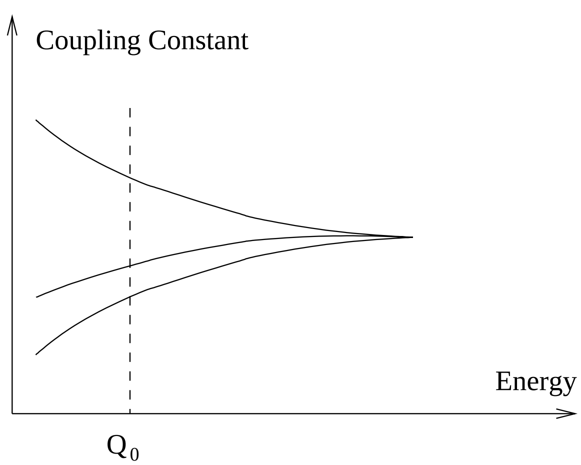

Figure 3.1: Coupling Constant

Need substantial remaining. Q_0 = observation point (e.g. M_2).
Constraints: 3 observable $\rightarrow$ 2 quantities.

$$\frac{dg_i}{dt} = \beta_i(g_i) \tag{3.26}$$

$$\begin{aligned}
\frac{d\frac{1}{g_i^2}}{dt} &= -2\beta_i^\circ && (3.27)\\
\frac{1}{g_3^2(Q_0)^2} - \frac{1}{g_3^2(M_{un.})} &= 2\beta_3^\circ \ln\frac{M_U}{Q_0} && (3.28)\\
\frac{1}{g_i(Q_0)^2} - 2\beta_i^\circ \ln\frac{M_U}{Q_0} &: \quad \textit{independent of } i && (3.29)\\
\frac{\frac{1}{g_i(Q_0)^2} - \frac{1}{g_j(Q_0)^2}}{\beta_i^\circ - \beta_j^\circ} &: \quad \textit{independent of } i, j && (3.30)
\end{aligned}$$

N.B.: of course

$$g_1^2 = \frac{5}{3} g'^2 \tag{3.31}$$

Master formula:

$$\beta_0 = -\frac{11}{3} e_2 + \frac{4}{3} T_{\frac{1}{2}} + \frac{1}{3} T_0 \begin{cases} real \times \frac{1}{2} \\ weyl \times \frac{1}{2} \end{cases} \tag{3.32}$$

Minimal Standard Model (MSM):

$$\begin{aligned}
\beta^{(3)} &= -11 + \frac{4}{3} \times 6 \times \frac{1}{2} = -7 && (3.33)\\
\beta^{(2)} &= -\frac{11}{3} \times 2 + \frac{4}{3} \times 12 \times \underbrace{\frac{1}{2}}_{weyl} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = -\frac{19}{6} && (3.34)\\
\beta^{(1)} &= \frac{3}{5}\{\frac{4}{3} \times \underbrace{3}_{families} \times \underbrace{\frac{1}{2}}_{weyl} [6 \times (\frac{1}{6})^2 + 3 \times (\frac{2}{3})^2 + 3(\frac{1}{3})^2 + 2 \times (\frac{1}{2})^2 + 1^2] \\
&\quad + \frac{1}{3} \times 2 \cdot (\frac{1}{2})^2\} = \frac{41}{10} && (3.35)
\end{aligned}$$

MSSM:

$$\Delta\beta^{(3)} = \frac{4}{3} \times 3 \times \underbrace{\frac{1}{2}}_{real} + \frac{1}{3} \times 12 \times \frac{1}{2} = 4 \tag{3.36}$$

2 Higgs + Higgsions:

$$\Delta\beta^{(2)} = \begin{array}{ll} \frac{4}{3} \times 2 \times \frac{1}{2} & \textit{Gaugions} \\ +\frac{4}{3} \times 2 \times \frac{1}{2} \times \frac{1}{2} & \textit{Higgsions} \\ +\frac{1}{3} \times 12 \times \frac{1}{2} & \textit{Sferminos} \\ +\frac{1}{3} \times \frac{1}{2} & \textit{Extra Higgs} \end{array}$$

$$= \frac{25}{6} \tag{3.37}$$

Higgsions:

$$\begin{aligned}\Delta\beta^{(1)} &= \frac{3}{5}\{\frac{4}{3} \times 2 \times (2 \times (\frac{1}{2})^2) \times \underbrace{\frac{1}{2}}_{weyl} \\ &+\frac{1}{3} \times 3[6 \times (\frac{1}{6})^2 + 3 \times (\frac{2}{3})^2 + 3 \times (\frac{1}{3})^2 + 2 \times (\frac{1}{2})^2 + 1] \\ &+\frac{1}{3} \times (\frac{1}{2})^2\} \\ &= \frac{5}{2}\end{aligned} \tag{3.38}$$

Leaving out Higgs:
MSM:

$$\beta^{(3)} = -7 \tag{3.39}$$

$$\beta^{(2)} = -\frac{10}{3} \tag{3.40}$$

$$\beta^{(1)} = 4 \tag{3.41}$$

MSSM:

$$\Delta\beta^{(3)} = 4 \tag{3.42}$$

$$\Delta\beta^{(2)} = \frac{10}{3} \tag{3.43}$$

$$\Delta\beta^{(1)} = 2 \tag{3.44}$$

$$\frac{\Delta\beta^{(3)} - \Delta\beta^{(2)}}{\beta^{(3)} - \beta^{(2)}} = -\frac{2}{11} \tag{3.45}$$

$$\frac{\Delta\beta^{(3)} - \Delta\beta^{(1)}}{\beta^{(3)} - \beta^{(1)}} = -\frac{2}{11} \tag{3.46}$$

$$\frac{\Delta\beta^{(2)} - \Delta\beta^{(1)}}{\beta^{(2)} - \beta^{(1)}} = -\frac{2}{11} \tag{3.47}$$

Therefore, all correlations to unification come from one doublet $\Rightarrow$ 6 effective doublets.

Also change in M_U:

$$\ln \frac{M_U}{Q} \propto \frac{1}{\beta_i - \beta_j} \tag{3.48}$$

$$(\frac{1}{\beta_i - \beta_j})_{SUSY} \simeq (\frac{1}{\beta_i - \beta_j})_{MSM}(1 - \frac{2}{11}) \simeq \frac{11}{9}(\frac{1}{\beta_i - \beta_j})_{MSM} \tag{3.49}$$

$$\frac{10^{17}}{10^2} \rightarrow 10^{12 \cdot \frac{11}{9} + 2} \simeq 10^{16.5} \tag{3.50}$$

3.2 $SU(5)$

- Fermion Multiplets

$$SU(3)\times SU(2)\times U(1)\subset SU(5) \tag{3.51}$$

Needs $U(1)$ traceless.

LH fields:

$$\begin{array}{c|ccccc} & 3 & 2 & 1 & d & \sum Y \\ \hline Q_L & 3 & 2 & \frac{1}{6} & 6 & 1 \\ CU_R & \bar{3} & 1 & -\frac{2}{3} & 3 & -2 \\ CD_R & \bar{3} & 1 & \frac{1}{3} & 3 & 1 \\ L_L & 1 & 2 & -\frac{1}{2} & 2 & -1 \\ CE_R & 1 & 1 & 1 & 1 & 1 \end{array} \tag{3.52}$$

Look for $5, 10$ with $\sum Y = 0$.

$$F_\mu : \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} & -\frac{1}{2} \\ & Cd_R & & e & \nu \end{pmatrix} \tag{3.53}$$

$T^{\mu\nu}$: antisymmetric

$$\begin{array}{ccccc} T^{\alpha\beta} & : & \bar{3}, & 1, & -\frac{2}{3} \\ T^{\alpha i} & : & 3, & 2, & \frac{1}{6} \\ T^{ij} & : & 1, & 1, & 1 \end{array} \tag{3.54}$$

- Symmetry Breaking

Adjoint (traceless)

$$\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} M \tag{3.55}$$

$$SU(5)\to SU(3)\times SU(2)\times U(1) \tag{3.56}$$

$$\left(\begin{array}{c|c} SU(3) & \\ \hline & SU(2) \end{array}\right) \tag{3.57}$$

Still need $SU(2)\times U(1)\Rightarrow U(1)$.

Minimal:

$$5\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{3.58}$$

There could be others.

Mass terms:

$$\varphi^{\mu}T^{\nu\rho}T^{\sigma\tau}\epsilon_{\mu\nu\rho\sigma\tau} \tag{3.59}$$

Note on "Majorana" mass terms:

$$\begin{aligned}
\Psi'\Psi &= (e\bar{\Psi}')\Psi && (3.60)\\
\Psi_L &= (\frac{1-\gamma_5}{2})\Psi && (3.61)\\
(\frac{1-\gamma_5}{2})\Psi_L &= 0 && (3.62)\\
\gamma_5 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} && (3.63)\\
\Psi_L &= \begin{pmatrix} \eta \\ -\eta \end{pmatrix} && (3.64)\\
C\Psi' &= \gamma_2 \Psi^{'*} && (3.65)\\
C\Psi_2' &= \begin{pmatrix} i\sigma_2\eta^* \\ i\sigma_2\eta^* \end{pmatrix} && (3.66)\\
(e\bar{\Psi}') &= \begin{pmatrix} i\sigma_2\eta' & i\sigma_2\eta' \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \eta \\ -\eta \end{pmatrix} && (3.67)\\
&\propto \eta_a'\epsilon^{ab}\eta_b && (3.68)
\end{aligned}$$

where, a and b are Dirac indices.

Therefore,

$$\eta_a^{(i)'}\epsilon^{ab}\eta_b^{(j)} \tag{3.69}$$

Note symmetric in $i \Leftrightarrow j$, due to Fermi statistics.

Link between texture and representation at unification: (g_{ab} symmetric)

$$\varphi^\mu T^{(a)\nu p} T^{(b)\sigma\tau} \tag{3.70}$$

allows self-mass for 2 component neutral fields.

$$\frac{\upsilon}{\sqrt{2}} \epsilon_{5\alpha\beta\gamma ji} \underbrace{T^{\alpha\beta} T^{\gamma i}}_{CU_R U_L} \tag{3.71}$$

$\varphi^*_\mu T^{\mu\nu} F_\nu$:

$$\frac{\upsilon}{\sqrt{2}} T^{5\alpha} F_\alpha \quad : \quad D_L \; CD_R \tag{3.72}$$

$$\frac{\upsilon}{\sqrt{2}} T^{54} F_4 \quad : \quad CE_R \; E_L \tag{3.73}$$

- B Validation

  Vector bosons:

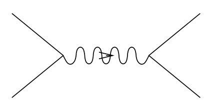

Figure 3.2: Vectro Bosons

$$T^*_{12} T^{14} X^2_4 X^4_2 F^{*2} F_4 \tag{3.74}$$

$$u^3_R u'_L \quad d^2_R e_L \tag{3.75}$$

Triplet Higgs:

$$(\epsilon_{12345} T^{12} T^{45} \varphi^3)(\varphi^*_3 T^{31} F_1) \tag{3.76}$$

$$\bar{u_{R_3}} \bar{e_R} \bar{u_{R_2}} \bar{d_{R_1}} \tag{3.77}$$

Single appearance of $\epsilon_{\alpha\beta\gamma}$.

Phenomenology $\Rightarrow M$ large.

- Implementing SB; Hierarchy problem $\varphi^+\varphi$, $\varphi^+ A\varphi$, $\varphi^+ A^2\varphi$, $trA^2\varphi^+\varphi$, trA^2, trA^3, trA^4, $tr(A^2)^2$. Need big vev for A, small for φ. Heavy φ^α not by decoupling, but by conspiracy. Not inconsistent, but ugly.

- Normalization

$$g\begin{pmatrix} \cdots & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix} \tag{3.78}$$

$$g'\begin{pmatrix} \frac{1}{3} & & & & \\ & \frac{1}{3} & & & \\ & & \frac{1}{3} & & \\ & & & -\frac{1}{2} & \\ & & & & -\frac{1}{2} \end{pmatrix} = g()_{\sum d_i^2=\frac{1}{2}} \tag{3.79}$$

$$g'^2(3\cdot(\frac{1}{3})^2+2\cdot(\frac{1}{2})^2) = g^2\frac{1}{2} \tag{3.80}$$

$$g'^2(\frac{5}{6}) = g^2\frac{1}{2} \tag{3.81}$$

$$g'^2 = \frac{3}{5}g^2 \tag{3.82}$$

$$\sin^2\theta_w = \frac{g'^2}{g^2-g'^2} = \frac{\frac{3}{5}g^2}{(1+\frac{3}{5})g^2} = \frac{3}{8} \tag{3.83}$$

Expct. ≈ 0.22.

Also, of course,

$$\frac{g_{SU(2)}}{g_{SU(3)}} = 1 \tag{3.84}$$

3.3 $SO(10)$ Unification

$SU(6)$?

$$\frac{6\times 5}{2}, F^{ab}, T^{\mu\nu} \tag{3.85}$$

$SU(5)$ in $SO(10)$: 5 complex components

$$Z_j = X_j + iY_j \tag{3.86}$$

$$< Z'|Z > = \underbrace{\sum X'_j X_j + Y'_j Y_j}_{SO(10)\ leaves\ this\ part\ invariant} + i \underbrace{\sum X'_j Y_j - Y'_j X_j}_{SP(10)\ leaves\ this\ part\ invariant} \tag{3.87}$$

$SO(10)$ commutators (structure contents): rotation in kl plane

$$\delta_\epsilon^{kl} X_j = \epsilon(\delta_{jk} X_l - \delta_{jl} X_k) \tag{3.88}$$

$$\begin{aligned}(\delta_\epsilon^{kl}\delta_\eta^{mn} - \delta_\eta^{mn}\delta_\epsilon^{kl})X_j &= \eta\delta_\epsilon^{kl}(\delta_{jm}X_n \\ &\quad -\delta_{jn}X_m) - \epsilon\delta_\eta^{mn}(\delta_{jk}X_l - \delta_{jl}X_k) \qquad (3.89)\\ &= \epsilon\eta\{\delta_{jm}\delta_{nk}X_l - \delta_{jm}\delta_{nl}X_k \\ &\quad -\delta_{jn}\delta_{mk}X_l + \delta_{jn}\delta_{ml}X_k \\ &\quad -\delta_{jk}\delta_{lm}X_n - \delta_{jk}\delta_{ln}X_m \\ &\quad +\delta_{jl}\delta_{km}X_n - \delta_{jl}\delta_{kn}X_m\} \qquad (3.90)\\ &= \delta_{nk}T^{lm} - \delta_{nl}T^{km} - \delta_{mk}T^{ln} + \delta_{ml}T^{kn} \qquad (3.91)\end{aligned}$$

Γ matrices "=" $\sqrt{rotation}$ (spinor rep.)

$$\{\Gamma_k, \Gamma_l\} = 2\delta_{kl} \tag{3.92}$$

Claim: $-\frac{1}{4}[\Gamma_k, \Gamma_l]$ satisfy the $SO(10)$ commutators.

$$[\Gamma_k, \Gamma_l] = 2(\Gamma_k\Gamma_l - \delta_{kl}) \tag{3.93}$$

So

$$\begin{aligned}\frac{1}{16}[[\Gamma_k, \Gamma_l], [\Gamma_m, \Gamma_n]] &= \frac{1}{4}[\Gamma_k\Gamma_l, \Gamma_m\Gamma_n] \qquad (3.94)\\ &= \frac{1}{4}(\Gamma_k\Gamma_l\Gamma_m\Gamma_n - \Gamma_m\Gamma_n\Gamma_k\Gamma_l) \qquad (3.95)\end{aligned}$$

Now use

$$\Gamma_a \Gamma_b = -\Gamma_b \Gamma_a + 2\delta_{ab} \tag{3.96}$$

to pull through T_k and T_l in turn. End terms cancel.
Pick up terms:

$$\frac{1}{4}(\Gamma_k \Gamma_l \Gamma_m \Gamma_n - \Gamma_m \Gamma_n \Gamma_k \Gamma_l) = \frac{1}{2}(-\delta_{nk}\Gamma_m \Gamma_l + \delta_{mk}\Gamma_n \Gamma_l - \delta_{nl}\Gamma_k \Gamma_m + \delta_{ml}\Gamma_k \Gamma_n) \tag{3.97}$$

Now use

$$\Gamma_m \Gamma_l = \frac{1}{2}([\Gamma_m, \Gamma_l] + 2\delta_{ml}) \quad etc. \tag{3.98}$$

$\delta\delta$ terms all cancel:

$$\begin{aligned} &\frac{1}{2}(-\delta_{nk}\Gamma_m \Gamma_l + \delta_{mk}\Gamma_n \Gamma_l - \delta_{nl}\Gamma_k T_m + \delta_{ml}\Gamma_k \Gamma_n) = \\ &\frac{1}{4}(-\delta_{nk}[\Gamma_m, \Gamma_l] + \delta_{mk}[\Gamma_n, \Gamma_l] - \delta_{nl}[\Gamma_k, \Gamma_m] + \delta_{ml}[\Gamma_k, \Gamma_n]) \end{aligned} \tag{3.99}$$

Compare to

$$\delta_{nk}\Gamma^{lm} - \delta_{nl}\Gamma^{km} - \delta_{mk}\Gamma^{ln} + \delta_{ml}\Gamma^{kn} \tag{3.100}$$

QED.

$$U^{-1}(R) T^\mu U(R) = R^\mu_\nu T^\nu \tag{3.101}$$

Construction of Γ matrices:

$$\begin{aligned} \Gamma_1 &= \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 && (3.102) \\ \Gamma_2 &= \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 && (3.103) \\ \Gamma_3 &= \sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 && (3.104) \\ \Gamma_4 &= \sigma_3 \otimes \sigma_2 \otimes 1 \otimes 1 \otimes 1 && (3.105) \\ \Gamma_5 &= \sigma_3 \otimes \sigma_2 \otimes \sigma_1 \otimes 1 \otimes 1 && (3.106) \\ &\vdots \end{aligned}$$

with Pauli σ – matrices does the job.
Note:

$$R_{12} = \frac{i}{2}\sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \quad (3.107)$$

$$R_{34} = \frac{i}{2} 1 \otimes \sigma_3 \otimes 1 \otimes 1 \otimes 1 \quad (3.108)$$

$$\vdots$$

So we diagonalize:

$$SO(2) \otimes SO(2) \otimes SO(2) \otimes SO(2) \otimes SO(2) \subset SO(10) \quad (3.109)$$

This gives us a $2^5 = 32$ – dimensional representation of $SO(10)$ by

$$R(e^{i\theta_{ab}T_{ab}}) = e^{i\theta_{ab}(-\frac{1}{4}[\Gamma_a,\Gamma_b])} \quad (3.110)$$

It is not quite irreducible.
Note

$$K = -i\Gamma_1\Gamma_2 \cdots \Gamma_{10} \quad (3.111)$$

anticommutes with all the Γ_i.
Also $K^* = K$ and K is Hermitean (exercise).

$$K^2 = -i\Gamma_i \cdots \Gamma_{10}\Gamma_i \cdots \Gamma_{10} = \underbrace{(-)^{1+\frac{10\times 9}{2}}}_{\textit{pulling through}} = 1 \quad (3.112)$$

Therefore, the R_{kl} commute with K and we can project spinos.

$$S \to \frac{1+K}{2} S \quad (3.113)$$

These make the $16 + 16'$ representations, which are irreducible.

Shift-register rotation: The components of S can be labeled by their $SO(2)^5$ eigenvalues $\pm\frac{1}{2}$ (supplying a $-i$).

$$R_{before} = iR_{standard} \quad (3.114)$$

Since

$$k = +\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \quad (3.115)$$

the 16 has even number of signs, $\overline{(16)}$ has odd number of signs.
Back to $SU(5)$, $v'Jv$ invariant, with

$$J = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} = T_{12} + \cdots + T_{910} \quad (3.116)$$

$$\upsilon \rightarrow \upsilon + \epsilon G\upsilon \quad (3.117)$$

$$\Delta\upsilon' J\upsilon = \epsilon\upsilon'(G^T J + JG)\upsilon \quad (3.118)$$

$$= \epsilon\upsilon'(-GJ + JG)\upsilon \quad (3.119)$$

$$\Rightarrow_{spinor} \sum \sigma_2 \ invariant \quad (3.120)$$

Similarly we identify

$$SU(3) \subset SO(6) \quad (3.121)$$

$$SU(2) \subset SO(4) \quad (3.122)$$

and an extra $U(1)$ for hyperchrnge. The diagonal elements (maximal torus) are simply represented as the $R_{2i-1,2i}(-trace\ part)$. Thus,

$$R_{12} - \frac{1}{3}(R_{12} + R_{34} + R_{56})\ etc. \in SU(3) \quad (3.123)$$

$$R_{78} - R_{910} \in SU(2) \quad (3.124)$$

With

$$\frac{1}{6}(R_{12} + R_{34} + R_{56}) - \frac{1}{4}R_{78} - R_{910} \propto U(1)_Y \quad (3.125)$$

Analysis of 16 standard model $Q - \#s$:
$5 + signs$

$$1\ state$$
$$| + + + + + >$$
$$SU(3) \times SU(2) \times U(1)\ singlet \quad (3.126)$$

$1 + signs$

$$5\ state, 2\ types$$
$$| - - + - - >$$

$$\begin{aligned}
&\underbrace{\begin{matrix}|-+---> \\ |+----> \end{matrix}} \\
&SU(3)\ \overline{(3)} \\
&SU(2)\ singlet \\
\tilde{Y} &= \frac{1}{6}(-1) - \frac{1}{4}(-2) = \frac{1}{3} \qquad (3.127) \\
&\underbrace{\begin{matrix}|---+-> \\ |----+> \end{matrix}} \\
&SU(3)\ singlet \\
&SU(2)\ doublet \\
\tilde{Y} &= \frac{1}{6}(-3) = -\frac{1}{2} \qquad (3.128)
\end{aligned}$$

With $Y = \tilde{Y}$, these can be identified as

$$d_R^C, \begin{pmatrix} -e \\ \nu \end{pmatrix}_L \qquad (3.129)$$

They are our $\bar{5}$ of $SU(5)$.
$3 + signs$

$$\begin{aligned}
&10\ state, 3\ types \\
&|+++--> \\
&SU(3)\ singlet \\
&SU(2)\ singlet \\
Y &= \frac{1}{6}(+3) - \frac{1}{4}(-2) = 1 \equiv e_R^C \qquad (3.130) \\
&\underbrace{\begin{matrix}|+--++> \\ |-+-++> \\ |+--++> \end{matrix}} \\
&SU(3)\ \bar{3} \\
&SU(2) singlet \\
Y &= \frac{1}{6}(-1) - \frac{1}{4}(2) = -\frac{2}{3} \equiv u_R^C \qquad (3.131) \\
&|++-+-> \\
&|+-++->
\end{aligned}$$

$$\begin{aligned} & |-+++-> \\ & |++--+> \\ & |+-+-+> \\ & |-++-+> \\ & SU(3)\ 3 \\ & SU(2)\ 2 \\ Y = \frac{1}{6}(1) = \frac{1}{6} \equiv & \begin{pmatrix} u \\ d \end{pmatrix}_L \end{aligned} \tag{3.132}$$

Comments:

1. The construction of Γ –matrices – anticommuting objects – used here contains the existance of bosonization in $1+1d$ field theories. Also, the fermionization of spin chains (from $\sigma \otimes \sigma \otimes \cdots \rightarrow$ anticommuting quantities), known as Jordan-Wigner trick.

2. $$|+++++> \equiv N_R^C \tag{3.133}$$

   plays an important role in current thinking about symmetric masses. It can get a mass (Majorana Mass) of the type

   $$M_{ij} \bar{N_R^i} N_R^{Cj} \tag{3.134}$$

   This involves breaking $SO(10)$, but not $SU(3) \times SU(2) \times U(1)$, so M can be large. Also, N_S can connect to ordinary left-handed neutrinos through the ordinary Higgs doublet, in the form

   $$\mu_{ij} \bar{N_R^i} L^{\mu j} \varphi_\mu^* \tag{3.135}$$

   where, i, j are formly indices and $\mu = SU(2)$ index.

   By 2^{nd} order perturbation theory we induce Majorana Masses for the ν_L.

   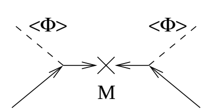

   Figure 3.3: Majorana Masses

   $$m \sim \frac{\mu^2}{M} \tag{3.136}$$

Breaking Scheme: Higgs φ in 16

$$\begin{aligned} < \varphi_N > &\neq 0 && (3.137) \\ SO(10) &\rightarrow SU(5) && (3.138) \end{aligned}$$

with "right" hyperchrnge.
Extra $U(1)$:

$$\begin{array}{lrr} N_R^C : & 5 & 5 \\ d_R^C, L_L : & -3 & -15 \\ u_R^C, Q_L, e_R^C : & 1 & 10 \end{array} \quad (3.139)$$

$$\alpha B + \beta L + \gamma Y \quad (3.140)$$

N_R^C:

$$\begin{aligned} -\beta &= 5 \\ \beta &= -5 && (3.141) \end{aligned}$$

d_R^C:

$$\begin{aligned} -\frac{\alpha}{3} + \frac{\gamma}{3} &= -3 \\ \alpha &= 5 && (3.142) \end{aligned}$$

L_L:

$$\begin{aligned} \beta - \frac{\gamma}{2} &= -3 \\ \gamma &= -4 && (3.143) \end{aligned}$$

u_R^C:

$$\begin{aligned} -\frac{5}{3} - \frac{2\gamma}{3} &\stackrel{?}{=} 1 && (3.144) \\ -\frac{5}{3} + \frac{\gamma}{3} &= 1 && (3.145) \end{aligned}$$

Q_L:

$$\begin{aligned}\frac{\alpha}{3}+\frac{\gamma}{6} &= \frac{5}{3}+\frac{2}{6}\\ &= 1\end{aligned}\tag{3.146}$$

e_R^C:

$$-\beta+\gamma=1\tag{3.147}$$

$$\alpha B+\beta L+\gamma Y=5B-5L-4Y\tag{3.148}$$

There are $45_{adjoint}$ and 10_{vectro} as before.
Fermion masses: bilinears in φ

$$\Gamma_1 = \sigma_1\otimes 1\otimes\cdots\otimes 1\tag{3.149}$$
$$\Gamma_2 = \sigma_2\otimes 1\otimes\cdots\otimes 1\tag{3.150}$$
$$\vdots$$

φ^* transforms as $e^{[\Gamma_\mu,\Gamma_\nu]^*}$

$$\begin{aligned}(C\varphi^*)' &= Ce^{[\Gamma_\mu,\Gamma_\nu]^*}\varphi^*\\ &= e^{[\Gamma_\mu,\Gamma_\nu]^*}C\varphi^*\end{aligned}\tag{3.151}$$
$$C\Gamma_\mu^* = \pm\Gamma_\mu C\tag{3.152}$$
$$C = \Gamma_1\Gamma_3\Gamma_5\Gamma_7\Gamma_9\tag{3.153}$$

With + sign:

$$C=\sigma_1\otimes i\sigma_2\otimes\sigma_1\otimes i\sigma_2\otimes\sigma_1\tag{3.154}$$

symmetric and real.
Anticommute with K: Construct Majorana bilinears

$$\varphi\varphi\sim(\varphi^*C)^+\ tensor\ \overrightarrow{made}\ from\ \Gamma\ \varphi\tag{3.155}$$

to be consistent with projection add number of indices. Irreducible $\Rightarrow$ Totally antisymmetric.

$$16\times 16 = \underbrace{10_{vector}}_{S}+\underbrace{120_{3-tensor}}_{A}+\underbrace{126_{5-tensor:self-dual}}_{S}\tag{3.156}$$
$$16\times\overline{16} = 1_{scale}+45_{2-tensor}+210_{4-tensor}\tag{3.157}$$

Final comments on $SO(10)$ vs. $SU(5)$: 3 RH neutrinos vs. $SU(5)\times U(1)$. Charge quantization.

$$Q \propto \epsilon(B-L) \tag{3.158}$$

$p : 1+\epsilon$, $e : -1-\epsilon$, $n : \epsilon$, $v : -\epsilon$.

N.B.: mechanics of charge quantization. $n \to pe\bar{v}$.