Chapter 11

Lattice Gauge

As I have mentioned repeatedly, this is the ultimate definition of QCD. (For electroweak theory, there is no satisfactory non-perturbative definition). I also discussed before the process of dimensional transmutation, you should refer back to this after we have gone through the explicit construction of LGT.

Agenda here:

1. Formulation of pure gauge theory.
2. Formulation of fermion theory, doubling phenomenon.
3. Confinement in strong coupling

Euclideanize, introduces $4d$ cubic lattice. On links introduce (for QCD) $SU(3)$ matrices $\underbrace{U_{n_1,n_2}}_{site\ lables}$.

They should be thought of as parallel transporters, i.e., solution of the equation

$$\begin{aligned} \nabla_\mu U &= 0 && (11.1)\\ &= \partial_\mu U + igA_\mu U && (11.2)\\ U &= P\ (ordered\ integral) && (11.3) \end{aligned}$$

Thus, if, say,

$$\psi(x) = U(x, x_0)\psi_0 \tag{11.4}$$

then

$$\nabla_\mu \psi = 0 \tag{11.5}$$

Note that there is path-dependence in this definition of $\psi(x)$ and the parallelism is only along the path.

Let me emphesis, though, that this is only a mnemonic, for relating to the (formal) continuous theory. LGT itself does not know about A.

$$U_{n_1 n_2} = U^{-1}_{n_2 n_1} \tag{11.6}$$

We will enforce a gauge symmetry

$$U_{n_1 n_2} \to \Omega(n_1) U_{n_1 n_2} \Omega^{-1}(n_2) \tag{11.7}$$

with site-dependent $\Omega(n) \in SU(3)$. Thus, the symmetry group is $SU(3)^{Z^4}$

The simplest nontrivial local invariant is the trace around a plaquette

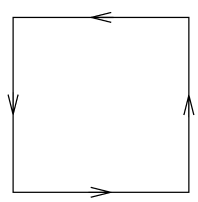

Figure 11.1: Plaquette.

$$tr U_{n,n+\hat{x}} U_{n+\hat{x},n+\hat{x}+\hat{y}} U_{n+\hat{x}+\hat{y},n+\hat{y}} U_{n+\hat{y},n} \equiv tr\Box_{n,xy} \tag{11.8}$$

For small fluctuations this is ≈ 3

This inspires the action

$$S = \frac{c}{g^2} \sum_{plaquettes} (3 - tr\Box) \tag{11.9}$$

where, $c = \frac{1}{24}$ to match continuous conventions.

To be completely explicit we should also specify the measure. It is the product of Haar measures $\prod_{links}[dU]$

$$[dU] = \int U^{-1} dU \underbrace{\wedge \cdots \wedge}_{8\ times} U^{-1} dU \tag{11.10}$$

$$= \int \underbrace{dx_i}_{\substack{group\ manifold \\ parametrization\ of}} \underbrace{det}_{jacobian} \underbrace{|| \frac{U^{-1}(x) dU(x)}{\partial x_i} ||}_{3\times 3\ traceless\ Hermitean} - use\ normalized\ basis \tag{11.11}$$

Crucial property is

$$[d(U_0U)] = [dU] \tag{11.12}$$

thus

$$\int [dU](non - singlet) = 0 \tag{11.13}$$

e.g.

$$\int [dU]U = \int [d(U_0U)]U \tag{11.14}$$
$$= \int [d\tilde{U}]U_0^{-1}\tilde{U} \tag{11.15}$$
$$= U_0^{-1}\int [dU]U \tag{11.16}$$
$$\int [dU]U = 0 \tag{11.17}$$

Note:

1. For small fluctuations, first non-trivial term is quadratic. Thus it must match continuous $\int trG_{\mu\nu}G_{\mu\nu}$
2. Everything is numerical – no units (→ dimensional transmutation).
3. Everything is finite.
4. Everything is algorithmic.
5. No gauge fixing required.

Fermions (quarks) live on sites they transform as

$$\psi_n \to U\psi_n \tag{11.18}$$

The simplest kinetic energy is

$$\frac{1}{2i}\sum_{n,unit\ displacement,\delta}\bar{\psi}_{n+\delta}\gamma_\delta U_{n+\delta,n}\psi_n \tag{11.19}$$

(i.e., $\gamma_{\hat{x}} = \gamma_1, \gamma_{-\hat{x}} = -\gamma_1, \cdots$)
Consider $U \approx 1$, plane wave

$$\psi_n \sim e^{ip.n}S \tag{11.20}$$

$(p.n = \underbrace{p_1 n_1}_{integer} + p_2 n_2 + p_3 n_3 + p_4 n_4)$.

We have

$$\frac{1}{2i} \sum_n (\bar{\psi}_{n+\hat{x}} - \bar{\psi}_{n-\hat{x}}) \gamma_{\hat{x}} \psi_n \to \sin p_x \gamma_{\hat{x}} \tag{11.21}$$

so inverse propagator $\gamma_i \sim p_i$ low energy states (= poles of propagator) for

$$\sum_i (\gamma_i \sin p_i)^2 = 0 \tag{11.22}$$

$$= \sum_i \sin^2 p_i \tag{11.23}$$

This occurs near $p_i \approx 0$ – smooth fields, but also for any $p_i = \pi$

Near $p_i = \pi$ the direction of E as p_i is reversed, so chirality is opposite.

This introduces 16 branches, where we wanted 1. You can scheme this effect but not eliminate it, by simple modifications. Recently more basic methods, involving adding considerable additional structure ($\approx$ extra dimensions, $4d$ domain wall) have emerged.

The brutal way (Wilson) is to add

$$-\frac{1}{2} \sum_{n,\delta} \bar{\psi}_{n+\delta} U \psi_n + \underbrace{4 \sum \bar{\psi}_n \psi_n}_{4 - \cos p_x - \cos p_y - \cos p_z - \cos p_\tau} \tag{11.24}$$

This eliminates the small energy at any $\cos p = -1$. It also, of course, violates chiral symmetry explicitly. The 4 must actually, be tuned, in an interaction dependent way, to get a light branch.

Massive quarks will stay put, so adding a massive quark-antiquark pair separated at distance R for time T inserts U matrices.

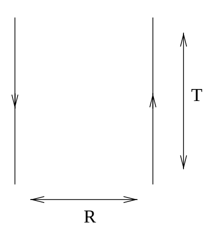

Figure 11.2: RT.

To determine the potential therefore we evaluate

$$e^{-V(R)T} = \lim_{T \to \infty} \frac{tr \int [dU] \int e^{-\frac{c}{g^2 \sum_{plaquette} \Box}}}{\int [dU] \int e^{-\frac{f}{g^2 \sum \Box}}} \tag{11.25}$$

In story coupling, expand the action

$$e^{-\frac{c}{g^2}\sum D} = \prod(1 - \frac{c}{g^2}\sum \square \cdots)) \tag{11.26}$$

The 1 term works fine in the denominator, but the integration over links appearing in the large loop will vanish. To get a non-zero answer we must pair these links.

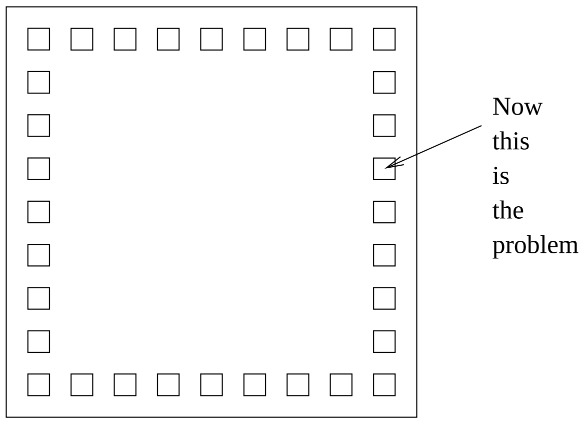

Figure 11.3: Sheet.

Keep going until the whole sheet is filled in. This gives a factor

$$e^{-V(R)T} \sim (\frac{1}{g^2})^{RT} \tag{11.27}$$

so $V(R) \propto R$.

Linear potential ($\Rightarrow$ confinement) is manifest at strong coupling. Of course, the continuous theory – fine lattice spacing – corresponds to weak coupling, as we saw earlier. If there is no phase transition, we get confinement there too. This has been shown numerically for QCD. QED ($U(1)$), on the other hand, has a phase transition.