12.4 Problem Set 4 Solutions

1. A nice derivation on the theoretical lower bound on Higgs mass (note experimental lower bound is $\mu_H \geq 120GeV$ by now) can be found in G. Attarelli, G. Isideri, Phys. Lett. B337 p141 ('94) (available on the SPIRES). This uses two–loop corrections and the result for ($\mu_t \simeq 174GeV$) as a function of cut-off scale Λ is as follows:

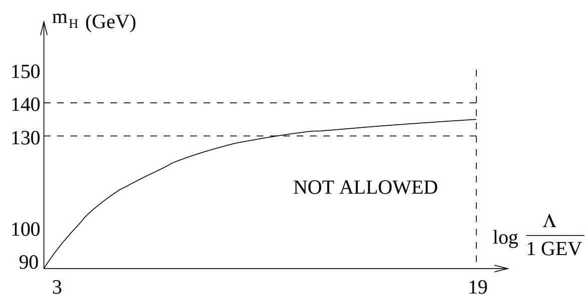

Figure 12.41: Lower Bound.

I will sketch the argument, see the paper for details.

Tree level Higgs potential is

$$V_{tree} = -\frac{\mu_0 \emptyset^2}{2} + \frac{\lambda_0 \emptyset^4}{24} \qquad (12.164)$$

where we fix λ_0 and μ_0 at renormalization scale $\mu_0 = \mu = v \simeq 245GeV$ (at weak scale).

As $\mu \to \mu + \delta\mu$, Higgs field strength scales as $\emptyset \to \emptyset(1 + \delta\eta)$ and $\lambda \to \lambda + \delta\lambda$.

Accordingly the renormalized potential is

$$V_{Ren} = \frac{1}{24}\lambda(\mu)((1 + \delta\eta(\mu))\emptyset)^4 \qquad (12.165)$$

For one–loop corrections we can write

$$1 + \delta\eta \simeq e^{\delta\eta} = exp[-\int_O^{\log\frac{\Lambda}{\mu_0}} \gamma[t']dt] \qquad (12.166)$$

where

$$\gamma \equiv -\frac{\mu}{\delta\mu}\delta\eta = -\frac{\delta\eta}{\delta \log\frac{\mu}{\mu_0}} \equiv -\frac{\delta\eta}{\delta t'} \qquad (12.167)$$

These one–loop effects can turn the tail of $V_{tree}(\emptyset)$ in downwards for large $\emptyset$ hence $V_{Ren}(\emptyset)$ may become unstable.

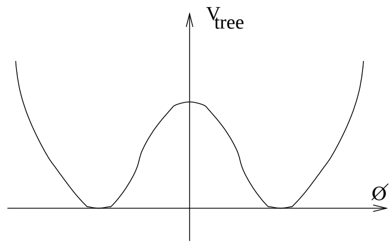

Figure 12.42: V_{tree}.

From Equation 12.165 the condition for stability is clearly

$$\lambda(\mu) > 0 \tag{12.168}$$

Thus we shall look at how λ evolves:

$$\frac{\lambda}{t} = \beta(t) = \frac{\partial}{\partial \ln \mu}(-\delta_\lambda + 2\delta_\emptyset) \tag{12.169}$$

where δ_λ is the counterterm for

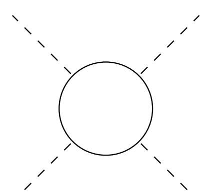

Figure 12.43: δ_λ Equation 12.169.

and $\delta_\emptyset$ is Higgs self–renormalization

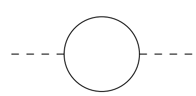

Figure 12.44: $\delta_\emptyset$ Equation 12.169.

We just draw contributing diagrams and give the result:

take only top quark.

$\rightarrow \delta_\lambda$ (from log–divergent pieces)

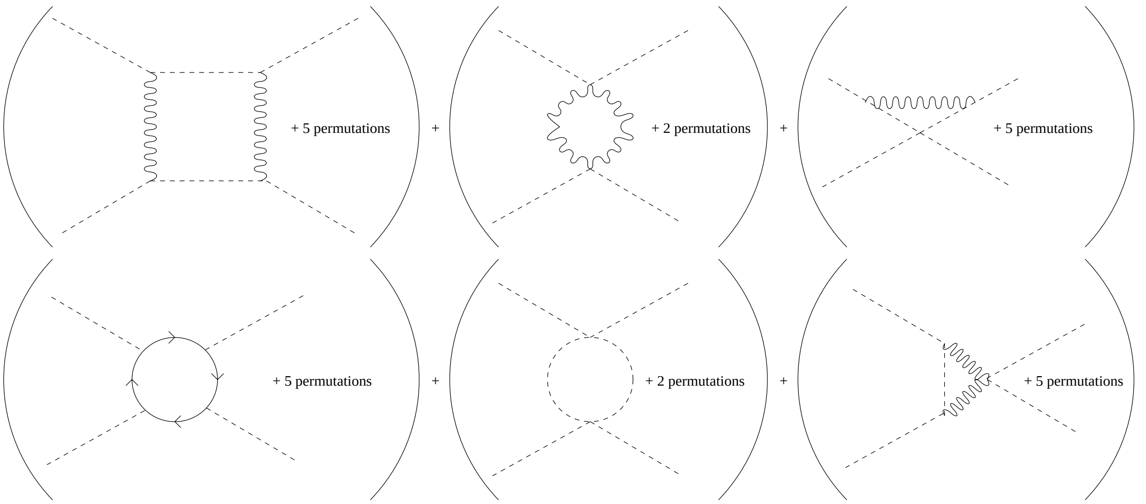

Figure 12.45: Contributing Diagrams δ_λ.

Figure 12.46: Contributing Diagrams $\delta_\emptyset$.

take only top.

$\rightarrow \delta_\emptyset$ (from log–divergent pieces)

Therefore,(see above mentioned paper) for the result. Actually I will quote two–loop result:

$$\frac{d\lambda}{dt} = \frac{1}{16\pi^2}[4\lambda^2 + 12\lambda g_t^2 - 36 g_t^4 - 9\lambda g_1^2 - 3\lambda g_2^2 + \frac{9}{2} g_1^2 g_2^2 + \frac{27}{4} g_1^4] \qquad (12.170)$$

where g_t is the Yukawa coupling for top quark, g_1 and g_2 are $U(1)_Y$ and $SU(2)_W$ coupling.

We have

$$g_t(\mu_0) = \frac{\sqrt{2} m_t}{V}(a + \delta_t(\mu_0)) \qquad (12.171)$$

$$\lambda_t(\mu_0) = \frac{3 m_H^2}{V^2}(a + \delta_\lambda(\mu_0)) \qquad (12.172)$$

at $\mu_0 = weak\ scale \simeq 100GeV$. (note that the funny factors of 3 in $\frac{\lambda}{m_H^2}$ is coming from my unusual V_{tree} definition).

Therefore, we see that λ can become negative, hence V_{Ren} unstable for particular values of Λ, m_t^2 and m_H^2. To find the range of m_H^2 for which $\lambda > 0$ fix $m_t^2 \simeq 174GeV$ (ignore QCD corrections). Then solve the above equation for $\frac{d\lambda}{dt}$ numerically and the one gets the range as shown in Figure 12.41.

2. Consider

$$L[\emptyset] = \frac{1}{2}\partial_\mu \emptyset \partial^\mu \emptyset - V(\emptyset) \tag{12.173}$$

Suppose we expand around $< \emptyset >= v(x)$ where $V(\emptyset)$ looks like

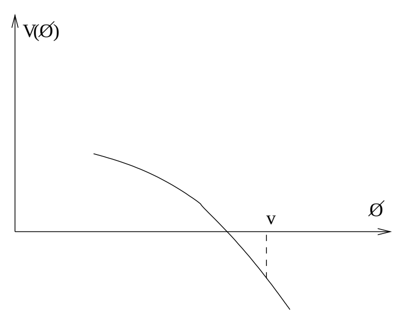

Figure 12.47: $V(\emptyset)$.

For simplicity lets take

$$V(\emptyset) \simeq \omega \emptyset^2 + C \tag{12.174}$$

with $\omega < 0$.

The ground state is a coherent state such that, for

$$\begin{aligned} \emptyset(\vec{x}) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2k_0}}(a_k e^{i\vec{k}\cdot\vec{x}} + a_k^+ e^{-i\vec{k}\cdot\vec{x}}) = \hat{\emptyset}_+ + \hat{\emptyset}_- & (12.175)\\ \hat{\emptyset}_+(\vec{x})|\xi> &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2k_0}}\xi(k)e^{i\vec{k}\cdot\vec{x}}|\xi> & (12.176)\end{aligned}$$

Lets consider a $0+1$ dimensional system for simplicity:

$$\hat{H} = \omega[a^t a + \frac{1}{2}] \quad (12.177)$$
$$< \xi|H|\xi > = ? \quad (12.178)$$

Solution to equation $a|\xi >= \xi|\xi >$ is $|\xi >= e^{\xi a^t}|o >$ in this case

$$< \xi|H|\xi >= \omega < o|e^{\xi a}(a^t a + \frac{1}{2})e^{\xi a^t}|o >= \omega(\xi^2 + \frac{1}{2})e^{\xi^2} \quad (12.179)$$

Consider an excitation:

$$< \xi|aHa^t|\xi >= \frac{d}{d\xi}\frac{d}{d\xi'} < \xi'|H|\xi > |_{\xi=\xi'} = \frac{d}{d\xi}\frac{d}{d\xi'}(\xi\xi' + \frac{1}{2})e^{\xi\xi'}\omega >< \xi|H|\xi > \quad (12.180)$$

Excitation get lower and lower energy (straightforward) since $\omega < 0$ (for classical unstable potential).

This reasoning is easily generalized to more non-trivial potentials which exhibit non-stable behavior classically and to $4D$ QFT, in which case

$$|\xi >= exp[\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2k_0}}\xi(k)e^{i\vec{k}\cdot\vec{x}}a^t_k]|0 > \quad (12.181)$$

3. (Thanks to Guide Festuccier) Observed value of

$$\frac{m_b}{m_\tau}|_{\mu_{\mu_z}} \simeq 1.62 \quad (12.182)$$

Both in supersymmetric and non-supersymmetric GUTs b and τ are in the same multiplet and get the same mass through Higgs coupling, hence

$$\frac{m_b}{m_\tau}|_{\mu_{\mu_{GUT}}} = 1 \quad (12.183)$$

To obtain the value at $\mu = \mu_z$ we should run the Yukawa couplings down to weak scale. One can ignore Yukawa couplings of other matter except top-quark:

MSSM:

$$\frac{d}{dt}\ln\lambda_t = \frac{1}{16\pi^2}[6\lambda_t^2 + \lambda_b^2 - \sum c_i g_i^2] \quad (12.184)$$

$$\frac{d}{dt}\ln\lambda_b = \frac{1}{16\pi^2}[-6\lambda_b^2 + \lambda_\tau^2 - \sum c'_i g_i^2] \quad (12.185)$$

$$\frac{d}{d\tau}\ln\lambda_t = \frac{1}{16\pi^2}[6\lambda_\tau^2 - 3\lambda_b^2 - \sum c''_i g_i^2] \quad (12.186)$$

$$t = \ln(\frac{\mu}{\mu_z}) \quad (12.187)$$

where g_i are $((U(1)$, $SU(2)$, and $SU(3))$ coupling for $i = 1, 2, 3$ respectively and the coefficients are:

MSSM:

$$c_i = (\frac{13}{5}, 3, \frac{16}{3}) \quad (12.188)$$

$$c'_i = (\frac{7}{15}, 3, \frac{16}{3}) \quad (12.189)$$

$$c''_i = (\frac{9}{5}, 3, 0) \quad (12.190)$$

SM:

$$c_i = (\frac{17}{20}, \frac{9}{4}, 8) \quad (12.191)$$

$$c'_i = (\frac{1}{4}, \frac{9}{4}, 8) \quad (12.192)$$

$$c''_i = (\frac{9}{4}, \frac{9}{4}, 0) \quad (12.193)$$

Subtract equations for λ_b and λ_τ and neglect λ_b with respect to λ_t to get

MSSM:

$$2\pi\frac{d}{dt}\ln(\frac{\lambda_b}{\lambda_\tau})^2 \simeq \frac{1}{4\pi}(\lambda_{t^2} - \frac{16}{3}g_3^2 + \frac{20}{15}g_1^2) \quad (12.194)$$

SM:

$$2\pi\frac{d}{dt}\ln(\frac{\lambda_b}{\lambda_\tau})^2 \simeq \frac{1}{4\pi}(\lambda_{t^2} - 8g_3^2 + 2g_1^2) \quad (12.195)$$

One straightforwardly gets

MSSM:

$$\frac{m_b}{m_\tau}(\mu_z) = (e^{-\frac{1}{16\pi^2}\int_0^{t(\mu_{GUT})}\lambda_t^2 dt'})(\frac{\alpha_3(\mu_z)}{\alpha_3(\mu_{GUT})})^{\frac{8}{9}} \tag{12.196}$$

(ignoring $O(g_1^2)$ with respect to $O(g_2^2)$).

SM:

$$\frac{m_b}{m_\tau}(\mu_z) = (e^{-\frac{1}{16\pi^2}\int_0^{t(\mu_{GUT})}\lambda_t^2 dt'})(\frac{\alpha_3(\mu_z)}{\alpha_3(\mu_{GUT})})^{\frac{4}{7}} \tag{12.197}$$

We have to obtain $\alpha_3 = \alpha_2 = \alpha_1|_{GUT}$ in MSSM vs. SM.

This is easily done by looking at one–loop running of MSSM and SM couplings and their unification at μ_{GUT}, see for instance R. Mohapatra hep-th 9801235 r2.

$$\alpha \equiv \frac{1}{(4\pi)^2}g^2 \tag{12.198}$$

$$\alpha_{GUT}^{MSSM} \simeq \frac{1}{24} \tag{12.199}$$

$$\alpha_{GUT}^{SM} \simeq \frac{1}{42} \tag{12.200}$$

One obtains

$$\frac{m_b}{m_z}(\mu_z)|_{MSSM} = exp(-\frac{1}{16\pi^2}\int \lambda_t^2)|_{MSSM} \times 2.56 \tag{12.201}$$

$$\frac{m_b}{m_z}(\mu_z)|_{SM} = exp(-\frac{1}{16\pi^2}\int \lambda_t^2)|_{SM} \times 2.51 \tag{12.202}$$

The difference mostly depends of the running of top-Yakawa coupling in MSSM vs. SM. According to R. H. Mohapatra, ("Suppersymmetry and Unification," Springer-Verlag, 2003) $\frac{m_b}{m_\tau} \simeq 3$ in SM which is bad as $\frac{m_b}{m_\tau} \simeq 1.62$ in reality and, $\frac{m_b}{m_\tau} \simeq 2.3$ in MSSM. But the latter result is model-dependent. Especially on the particular breaking mechanism.