12.2 Problem Set 2 Solutions

1. • I will use a basis m, which

$$\psi^C = i\gamma^2\psi^* = C\gamma^\circ\psi^* \tag{12.47}$$

We can define left (light) handed Majorana fields as,

$$\begin{aligned} \omega &= \psi_L + (\psi_L)^C & (12.48)\\ \chi &= \psi_R + (\psi_R)^C & (12.49) \end{aligned}$$

so that

$$\begin{aligned} \omega &= \omega^C & (12.50)\\ \chi &= \chi^C & (12.51) \end{aligned}$$

Note that

$$\begin{aligned} (\psi_L)^C &= (\psi^C)_R & (12.52)\\ (\psi_R)^C &= (\psi^C)_L & (12.53) \end{aligned}$$

Then

$$\begin{aligned} -\mu_R\overline{\psi}\psi &= -\mu_R(\overline{\psi_R}(\psi_R)^C + \overline{\psi_R^C}\psi_R) & (12.54)\\ -\mu_L\overline{\omega}\omega &= -\mu_L(\overline{\psi_L}(\psi_L)^C + \overline{\psi_L^C}\psi_L) & (12.55) \end{aligned}$$

are the right (left) handed mass terms for Majorana fields.

• Generalizing to N flavor, $i, j = 1, \cdots, N$, we have $-\mu_{ij}\overline{\psi^{Ci}}\psi^j + h.c$
Using above definition for C and anti-symmetry of Grassmann variables are sees that μ_{ij} can be taken as symmetric.

• For one flavor case general Dirac and Majorana mass is

$$\begin{pmatrix} \overline{\omega} & \overline{\chi} \end{pmatrix} \begin{pmatrix} -\mu_L & \frac{m}{2} \\ \frac{m}{2} & -\mu_R \end{pmatrix} \begin{pmatrix} \omega \\ \chi \end{pmatrix} \tag{12.56}$$

Since

$$\frac{m}{2}(\overline{\omega}\chi + \overline{\psi}\omega) = m(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L) = Dirac\ Mass \tag{12.57}$$

One can diagnolize this by a unitary transformation to get the eigenvalues (in case $\mu_L = 0$ for instance) $\mu = \mu_R, \frac{m^2}{\mu_R}$

- This is an example of the "see–saw" mechanism. Although one can not right down relevant (dimension ≤ 4) Majorana mess term in standard model, one can in some GUT's, e.g. $SO(10)$. Therefore, the natural Majorana mass $\sim O(10^{15}\ Gev)$, which is the GUT scale in typical theories.Then above mechanism would give a LH fermion with a tiny mass

$$\frac{m^2}{\mu} \sim \frac{(100\ Gev)^2}{10^{15}\ Gev} \sim 10^{-2}\ eV \tag{12.58}$$

which is consistent with observation of scalar neutrinos. Note that for the typical Dirac mass μ, I take $\sim 100\ Gev$ since they are obtained by Higgs much at weak scale.

- Same mechanism works for $N > 1$ flavors in which case one diagonalizes the general mass make it with a unitary transformation.

2. (a) Let's begin with listing the matter content of $S\mu$ indicating the hypercharges:

Quarks:

$$\rho_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}_{\frac{1}{6}}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}_{\frac{1}{6}}, \begin{pmatrix} t_L \\ b_L \end{pmatrix}_{\frac{1}{6}} \tag{12.59}$$

$$q^u_{R\ \frac{2}{3}} = u_R, c_R, t_R \tag{12.60}$$

$$q^d_{R\ -\frac{1}{3}} = d_R, s_R, b_R \tag{12.61}$$

Leptons:

$$l_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}_{-\frac{1}{2}}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}_{-\frac{1}{2}}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}_{-\frac{1}{2}} \tag{12.62}$$

$$l_{R\ -1} = l_R, \mu_R, \tau_R \tag{12.63}$$

Higgs:

$$\Phi = \begin{pmatrix} \emptyset^+ \\ \emptyset_\circ \end{pmatrix}_{\frac{1}{2}} \tag{12.64}$$

We want to find a term with $B \neq 0$. Clearly this term should involve quarks. The restrictions are: $SU(3)$ color, $SU(2)$ weak, $U(1)_Y$ and Lorentz invariance. (l, ρ, τ separately are not fundamental). $SU(3)$ requires three combinations:

$$3 \otimes 3 \otimes 3 = 1 + 8 + 8 + 10 \tag{12.65}$$

$$3^* \otimes 3^* \otimes 3^* = 1 + 8^* + 8^* + 10^* \tag{12.66}$$

$$3 \otimes 3^* = 1 + 8 \tag{12.67}$$

Last one can not violate B hence discarded. First two are $\rho\rho\rho$ (or $\overline{\rho^c\rho^c\rho^c}$) and $\overline{\rho\rho\rho}$ (or $\rho^c\rho^c\rho^c$) which are not Lorentz invariant unless we include another fermion which should be a lepton in order not to spoil $SU(3)_{color}$. Hence the lowest dimensional operators which has $B \neq 0$ are $\sim \rho\rho\rho l$ with dimension 6.

(b) There are four types: $\rho\rho\rho l$, $\rho^*\rho^*\rho^* l^*$, $\rho\rho\rho l^*$, $\rho^*\rho^*\rho^* l$. First two does not violate $B - L$, last two does. This problem amounts to see that last two are in violation of at last one of the after mentioned symmetries of $S\mu$. Consideration of $\rho\rho\rho l^*$ is sufficient:

- To have $SU(2)$ invariance we need even number of left handed: $\rho_L\rho_L\rho_L l_L^*$, $\rho_L\rho_R^u\rho_R^u l_L^*$, $\rho_L\rho_R^u\rho_R^d l_L^*$, $\rho_L\rho_R^d\rho_R^d l_L^*$, $\rho_R^u\rho_R^u\rho_R^u l_R^*$, $\rho_R^u\rho_R^u\rho_R^d l_R^*$, $\rho_R^u\rho_R^d\rho_R^d l_R^*$, $\rho_R^d\rho_R^d\rho_R^d l_R^*$.
- Note that ρ is either ρ or $\overline{\rho^c}$. l^* can only be a $\overline{l}$.
- None of the above can be Lorentz invariant hence it is impossible to violate $B - L$ with a dimension 6 operator.

Actually a generalization of above reasoning glons that $B - L$ can not be involved in $S\mu$ neither perturbatively nor non-perturbatively. However $B-L$ violation would after a nice explanation for observed baryon asymmetry in the universe. One nice feature of GUT's is that there are consistent GUT's with relevant $B - L$ violating terms (e.g. $SO(10)$).

(c) To violate L we need at least are lepton. If we insist to have only one lepton than we need to contract it with at least one quark. This would violate $SU(3)$ hence we need three quarks, but this term ($l\rho\rho\rho$) is dimension 6, no way. Consider two leptons, in order to violate L these should have same lepton number, hence Majorana type contraction: $\overline{l_L^c} l_L$ or $\overline{l_R^c} l_R$. But these have total hypercharge -1 and -2 each. To cancel this we are only left with ϕ's to add. Therefore, we get $\overline{l_L^c} l_L \phi^2$ with dimension 5 or $\overline{l_R^c} l_R \phi^4$ with dimension 7. As ϕ gets a VEV by Higgs

$$\overline{l_L^c} l_L \phi^2 \to \overline{l_L^c} l_L v^2 \tag{12.68}$$

becomes a Majorana mess term.

3. (a) $SU(2)$ in adjoint can be represented by

$$\tau^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \tag{12.69}$$

$$\tau^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} & -i & \\ i & & -i \\ & i & \end{pmatrix} \tag{12.70}$$

$$\tau^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix} \tag{12.71}$$

- To have a cross product representation under $SU(2) \times U(1)_Y$ all elements in the triplet should carry same hypercharge Y. Then the covariant derivative takes the form

$$\Delta_\mu = I_\mu + igA_\mu^a \tau^a + ig' B_\mu Y \tag{12.72}$$

where, $Y = \begin{pmatrix} Y & & \\ & Y & \\ & & Y \end{pmatrix}$

- We want to give a VEV to triplet Higgs

$$\phi^3 = \begin{pmatrix} \emptyset_1 \\ \emptyset_2 \\ \emptyset_3 \end{pmatrix} \tag{12.73}$$

such that only one of the linear combinations of generators τ', τ^2, τ^3, Y is unbroken. This will be the electric charge Q. This will be a diagonal $U(1)$, hence

$$Q = a\tau^3 + bY \tag{12.74}$$

overall constant can be observed into charge

$$Q = \tau^3 + bY \tag{12.75}$$

b is arbitrary but for convenience we take as 1

$$Q = \tau^3 + Y = \begin{pmatrix} Y+1 & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Y-1 \end{pmatrix} \tag{12.76}$$

This should have a zero eigenvalue, hence $Y \in \{+1, -1, 0\}$. However in case $Y = 0$ only τ' and τ^2 are broken by $\begin{matrix} 0 \\ Y \\ 0 \end{matrix}$, hence we get $SU(2) \times U(1) \to U(1) \times U(1)$. We should choose $Y \in \{+1, -1\}$.

(b) Consider both a doublet ϕ^2 with the covariant derivative

$$D_\mu \phi^2 = (I_\mu + ig\frac{\sigma^a}{2}A^*_\mu + ig'\frac{1}{2}B_\mu)\phi^2 \tag{12.77}$$

and a triplet with

$$D_\mu \phi^3 = (I_\mu + igA^*_\mu \tau^a + ig'YB_\mu)\phi^3 \tag{12.78}$$

Expanding out $|D_{\mu\phi^2}|^2 + |D_\mu \phi^3|^2$ for $|\phi^2|^2 = v_2^2$, $|\phi^3|^2 = v_3^2$ we get for $Y = \pm 1$:

$$m^2_{w\pm} = \frac{1}{4}g^2(v_2^2 + 2v_3^2) \tag{12.79}$$

$$m^2_z = \frac{1}{4}(g^2 + \rho'^2)(v_2^2 + 4v_3^2) \tag{12.80}$$

$$m_\gamma = 0 \tag{12.81}$$

If we keep ϕ^2 we have the option $Y = 0$ in contrast to above since ϕ^2 already breaks to $U(1)$. For this case, $Y = 0$:

$$m^2_{w\pm} = \frac{1}{4}(v_2^2 + 4v_3^2) \tag{12.82}$$

$$m^2_z = \frac{1}{4}(g^2 + \rho'^2)v_2^2 \tag{12.83}$$

$$m_\gamma = 0 \tag{12.84}$$

Note that m_z^2 is entirely coming from usual doublet Higgs.

- $\frac{v_3}{v_2}$ can be constrained as follows: See H. E. Haber, "Minimal and Non-minimal Higgs Bosons," in "Phenomenology of $S\mu$ and Beyond," D.P. Rey and P. Rey world scientific, 1989 (this is in library: QC793.W66 1989). An experimental fact that

$$\rho \equiv \frac{\mu_w^2}{\mu_z^2 \cos^2\theta_w} \tag{12.85}$$

is very close to 1:

$$\rho = 1 - \epsilon^2, 0 < \epsilon \ll 1 \tag{12.86}$$

On the other hand for a general Higgs content one can express ρ in term so f the casming of $SU(2)$ and $U(1)$ as:

$$\rho = \frac{\sum_{T,Y}(T(T+1) - Y^2)| < \phi_{T,Y} > |^2}{\sum_{T,Y} 2Y^2| < \phi_{T,Y} > |^2} \tag{12.87}$$

where T, Y denote the representation, $< \phi_{T,Y} >$ is the VEV of particular Higgs in the sum. Note that for $T = \frac{1}{2}$, $Y = \pm\frac{1}{2}$ one naturally gets $\rho = 1$ (for any number of Higgs fields with $T = 1$, $Y = \pm\frac{1}{2}$). For our problem we get

$$\rho = \frac{\frac{1}{2}v_2^2 + v_3^2}{\frac{1}{2}v_2^2 + 2v_3^2} = \frac{1 + 2(\frac{v_3}{v_2})^2}{1 + 4(\frac{v_3}{v_2})^2} = 1 - \epsilon^2 \tag{12.88}$$

Therefore, $\frac{v_3}{v_2}$ should be very small.

$$\rho \simeq 1 - 2(\frac{v_3}{v_2})^2 = 1 - \epsilon \Rightarrow \frac{v_3}{v_2} = \frac{\epsilon}{\sqrt{2}} \ll 1 \tag{12.89}$$

This shows that adding a new type of Higgs field to the usual doublet is highly constrained by experiments. However, one can clearly add any number of doublets without violating $\rho = 1 - \epsilon$ constraint. This possibly is explored in the next problem.

(c) Initially we have $6 + 4 = 10$ real D.O.F. 3 is eaten and we have left with 7 real scalar D.O.F. One of them is usual Higgs with $Q = 0$, isospin $-\frac{1}{2}$. The rest for $Y = \pm 1$ are two neutral scalars, two scalars of charge ± 1, two scalars of charge ± 2 as clear from above Q matrix.

(d) We now have the possibility of a Higgs field with $Y = \pm 1$. Recall from Problem 2 that the biggest constraint for a L–violating term was imposed by preserving hypercharge. Now we can write down $\overline{l_L^c} l_L \phi_{+1}^3$, which is dimension 4 hence marginal. Note however, that lepton number violating processes are quick constrained by experiments hence $\frac{v_3}{v_2}$ should again be very small in accord with our discussion in previous part.

4. (a) We have two Higgs doublets ϕ_1 and ϕ_2 with condensation

$$< \phi_1 >= \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \tag{12.90}$$

$$< \phi_2 >= \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \tag{12.91}$$

- Then the masses of the gauge fields are

$$m^2_{w\pm} = \frac{1}{4} g^2 (v_1^2 + v_2^2) \tag{12.92}$$

$$m^2_z = \frac{1}{4} (\rho^2 + \rho^{'2})(v_1^2 + v_3^2) \tag{12.93}$$

$$m_\gamma = 0 \tag{12.94}$$

Therefore,

$$\frac{m^2_w}{m^2_z} = \frac{g^2}{g^2 + g^{'2}} \tag{12.95}$$

is same as in the case of single doublet.

- Furthermore, using the general formula for ρ (from previous problem) we saw that

$$\rho = 1 = \frac{m^2_w}{m^2_z \cos^2 \theta_w} \tag{12.96}$$

Therefore, θ_w is also the same as before.

- Started with 8 real scalar D.O.F. 3 absorbed into congitiduval modes of gauge mesons $w\pm$ and z. Therefore, 5 left out of which one is the usual neutral Higgs: $\phi_1 = \begin{pmatrix} 0 \\ v_1 + h \end{pmatrix}$, 4 others are $\phi_2 = \begin{pmatrix} \alpha + i\beta \\ \gamma + i\delta \end{pmatrix}$, α and β are charge $+1(\alpha + i\beta)$, $-1(\alpha - i\beta)$, γ and δ are both charge zero.

(b) See H. E. Haber in QC793.W66 1989

(c) General Yukawa coupling to quarks reads

$$\lambda_1 \overline{q^\alpha_L} \phi^\alpha_1 q^d_R + \lambda_2 \epsilon_{\alpha\beta} \overline{q^\alpha_L} \phi^{*\beta}_1 q^u_d + \lambda_3 \overline{q^\alpha_L} \phi^\alpha_2 q^d_R + \lambda_4 \epsilon_{\alpha\beta} \overline{q^\alpha_L} \phi^{*\beta}_1 q^u_R + \tag{12.97}$$

However, under new $U(1)$, ϕ_1 has charge -1, ϕ_2 has $+1$, q_R^d, q_R^u has $+1$ then 2^{nd} and 3^{rd} terms not allowed.

- This means ϕ_1 cannot couple to q_d^u, ϕ_2 cannot couple to q_R^d as in part above.
- Extra restriction on the potential in the previous part is that $(\phi_1^+\phi_2)(\phi_2\phi_1^+)$ term is not allowed.
- From continuous $U(1)$ breaking we get an additional Goldstone boson, the axion. It is proportional to Tv_2, where T is the $U(1)$ generator that generates $\phi_1 \to \phi_1 e^{-i\lambda}$, $\phi_2 \to \phi_2 e^{i\lambda}$. Therefore, its coupling to quarks are

$$\epsilon_{\alpha\beta}\overline{q_L^\alpha}q_R^u\phi_2^{*\beta} \to (\overline{u_L}u_R\lambda_u + \overline{c_L}c_R\lambda_c + \overline{t_L}t_R\lambda_t)\underbrace{ax}_{axion} \tag{12.98}$$