Chapter 12

Problem Set Solutions

12.1 Problem Set 1 Solutions

1.

$$\vec{A} = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \tag{12.1}$$

(a)

$$\vec{\nabla}\times\vec{A} = \frac{1}{x^2+y^2}\nabla\times\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} + \nabla\frac{1}{x^2+y^2}\times\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = 0 \tag{12.2}$$

except at $x = y = 0$ where $\vec{\nabla}\times\vec{A}$ is singular.

(b) For any closed path which does not wind around $x = y = 0$ line one gets

$$\oint_C d\vec{S}\cdot\vec{A} = 0 \tag{12.3}$$

because of above.

If C winds around the $z - \alpha\times\beta$ one instead gets,

$$\oint_C d\vec{S}\cdot\vec{A} = 2\pi \tag{12.4}$$

We conclude that

$$\vec{\nabla}\times\vec{A} = 2\pi\delta(x)\delta(y)\hat{z} \tag{12.5}$$

A way to realize the setup is a thin long selonoid at $z - \alpha\times\beta$.

(c)

$$H = \frac{(\vec{p} + q\vec{A})^2}{2\mu} = \frac{\vec{p}^2 + 2q\vec{A}\vec{p} + q^2\vec{A}^2}{2\mu} \tag{12.6}$$

using $\vec{\nabla} \times \vec{A} = 0$.
We change to cylindrical cards and consider the wave-function

$$\psi(\rho, z > \emptyset) = \psi(\rho, z)e^{in\emptyset} \Rightarrow H = \frac{p_\rho^2}{2\mu} + \frac{p_z^2}{2\mu} + \frac{L_z^2}{2\mu\rho^2} + \frac{A_\emptyset L_z}{\mu} + \frac{q^2 A^2}{2\mu} \tag{12.7}$$

For $\vec{A} = \frac{\hat{u}}{q}$ one gets

$$H\psi = \frac{1}{2\mu}(p_\rho^2 + p_z^2 + \frac{1}{\rho^2}(\hbar + q)^2)\psi \tag{12.8}$$

We see the contrufugel pet.

$$V = \frac{1}{2\mu\rho^2}(\hbar + q)^2 \tag{12.9}$$

unless $q = \hbar m$ (in which case $\psi \to \psi e^{im\emptyset}$) we change the spectrum. For ψ to be single valued, $\emptyset = \frac{2\pi\hbar}{q} \Rightarrow$ flux quantization.

(d) As $\hbar \to 0$ the dependence on $\vec{A}$ of the spectrum gets away so this may quantum. As $\hbar \to 0$, $V = \frac{1}{2\mu\rho^2}q^2$. A classical effect is that as you change the strength of A, i.e., q spectrum changes continuously.

(e) If you Legendre transform you see that

$$d \supset \frac{1}{2}\mu\rho^2\emptyset^2 - \emptyset\rho q A_\emptyset \tag{12.10}$$

in cylindrical cards (there are other terms that L don't better). A conserved charge associated with the solutions around the $z - \alpha \times \beta$ is

$$d_z = \frac{\partial d}{\partial \emptyset} = m\rho^2\emptyset - \rho q A_\emptyset \tag{12.11}$$

This is canonical momentum. The mechanical momentum $L_z + \rho q A_\emptyset$ is not necessarily conserved.

2. (a)

$$A_{\mu\nu\rho} \to A_{\mu\nu\rho} + I_{[\mu\Lambda\nu\rho]} \tag{12.12}$$

Define

$$F_{\mu\nu\rho\sigma} = I_{[\mu\Lambda\nu\rho\sigma]} \tag{12.13}$$

Then

$$K.E. = -\frac{1}{2}\frac{1}{4!}{}_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} \tag{12.14}$$

E.O.M.:

$$I_\mu F^{\mu\nu\rho\sigma} = 0 \tag{12.15}$$

(Bronchi is trivial since there is no 5–index anti-symmetric tensor).
Easiest way to see the number of d.o.f. is to take Poincare dual:

$$F(x) = \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}(x) \tag{12.16}$$

so there is one field degree of freedom off-shell.
On-shell one uses E.O.M.:

$$I_\mu F(x) = 0 \Rightarrow F(x) = F = constant \tag{12.17}$$

This is suggested to be associated with the cosmological constant: $hep-th$ 0111032, $hep-th$ 0005276

(b) $A_{\mu\nu\rho}$ couples to volume-form $dx^\mu \Lambda dx^\nu \Lambda dx^\rho$. The classical source coupling is $\int A_{\mu\nu\rho} dx^\mu \Lambda dx^\nu \Lambda dx^\rho$. Under a g.t. this changes as

$$\int_\mu A_{\mu\nu\rho} dx^\mu \Lambda dx^\nu \Lambda dx^\rho \to \int_\mu A_{\mu\nu\rho} dx^\mu \Lambda dx^\nu \Lambda dx^\rho + \int_\mu I_\mu A_{\nu\rho} dx^\mu \Lambda dx^\nu \Lambda dx^\rho \tag{12.18}$$

We should require that

$$\int_{I_\mu} \Lambda_{\nu\rho} dx^\nu \Lambda dx^\rho = 0 \tag{12.19}$$

where I_μ denotes the boundary of the shape it couples to. So either you require $\Lambda_{\nu\rho}(I_\mu) = 0$ to be the only sensible g.t.'s or you require $I_\mu = 0$ (μ is compact) (which solves the problem for arbitrary $\Lambda_{\nu\rho}$).

(c) E.O.M. gets modified as

$$I_\mu F^{\mu\nu\rho\sigma} = J^{\nu\rho\sigma} \tag{12.20}$$

Let's find the source

$$J^{\mu\nu\rho}(x) = \frac{\delta}{\delta A^{\mu\nu\rho}(x)} \int_\mu A_{\alpha\beta\gamma}(y) dy^\alpha \Lambda dy^\beta \Lambda dy^\gamma \tag{12.21}$$

To vary with respect to $A^{\mu\nu\rho}(x)$ which lives on Minkewski, we should work out the embedding of μ into Minkewski. Parameterize the space-time coordinate on the world-volume as $y^\mu(u_1, u_2, u_3)$. Then above integral is

$$\int_{Minkewski} d^4x A_{\alpha\beta\gamma}(x) det(\frac{dy^\alpha}{dx^\mu})\delta(F(y)) \tag{12.22}$$

where $F(y)$ defines the surface

$$J_{\alpha\beta\gamma}(x) = \int \delta^4(x - y(u_1, u_2, u_3))(det\frac{Iy^\alpha}{du^i})d^3u \tag{12.23}$$

(d) Important features are:

- B_μ encodes all information in $A_{\mu\nu\rho}$.
- It has the gauge symmetry.
- $$B^\mu \to B^\mu = I_\nu \Lambda^{\mu\rho} \tag{12.24}$$
- $$F_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}\nabla \cdot B \tag{12.25}$$

(e) Complete solution can be found in Peskin and Schroeder.

3.

4. (a) To find a basis for $SU(N)$ matrices parameterize the $N \times N$ traceless and Hermitian matrix. In case of $SU(3)$ this is

$$\begin{pmatrix} a & b+ic & d+ie \\ b-ic & f & g+ih \\ d-ie & \rho-ih & -a-f \end{pmatrix} = a\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} + b\begin{pmatrix} & 1 & \\ 1 & & \\ & & 0 \end{pmatrix} + c\begin{pmatrix} & i & \\ -i & & \\ & & 0 \end{pmatrix} + d\begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix} +$$

$$e\begin{pmatrix} & & i \\ & 0 & \\ -i & & \end{pmatrix} + f\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} +$$
$$g\begin{pmatrix} 0 & & \\ & & 1 \\ & 1 & \end{pmatrix} + h\begin{pmatrix} 0 & & \\ & & i \\ & -i & \end{pmatrix} \quad (12.26)$$

We read of the basis elements $\tilde{T}_a$ as coefficient of $a, b, \cdots, h$, requiring $trw^a w^b = \frac{1}{2}\delta^{ab}$ means choosing $w^a = \frac{1}{2}\tilde{T}_a$. This is a nice basis.

(b)

$$|\alpha|^2 + |\beta|^2 = 1 = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 \quad (12.27)$$

Therefore, topology of $SU(2)$ is $S3$.

Topology of $SU(3)$ is an $S3$ bundle over $S5$ (see $hep - th$ 9812006).

(c) For any representation of a Lie algebra $[T^a, T^b] = if^{abc}T^c$ one can get a conjugate representation by $\tilde{T}^a = -T^a$ because taking complex conjugate of the commutation relations give $[-T^{a*}, -T^{b*}] = if^{abc}(-T^{c*})$ for f^{abc} real. Since T^a are Hermitian complex conjugate of a covariant vector transforms as contervariant vector.

A general tenser with n upper, m lower indices can be used to denote a general (might be reducible) representation: $\rho^{j_1\cdots j_n}_{i_1\cdots i_m}$ transfers as

$$\rho \rightarrow [T_a\rho]^{j_1\cdots j_n}_{i_1\cdots i_m} = \sum_{l=1}^{n}[T_a]^{j_l}_k \rho^{j_1\cdots k\cdots j_n}_{i_1\cdots i_m} - \sum_{l=1}^{m}[T_a]^k_{i_l}\rho^{j_1\cdots j_n}_{i_1\cdots k\cdots i_m} \quad (12.28)$$

From this transformation law, it is clear that one can impose symmetry among $(j_1 \cdots j_n)$ and $(i_1 \cdots i_m)$ and also one can impose tracelessness: $\delta^{i_1}_{j_1}\rho^{j_1\cdots j_n}_{i_1\cdots i_m} = 0$

In fact every tenser with n symmetric upper and m symmetric lower index with the additional restriction of tracelessness corresponds to an irreducible representations.

δ^j_i transforms

$$[T^a\delta]^i_j = [T^a]^k_j\delta^i_k - [T^a]^i_k\delta^k_j = 0 \quad (12.29)$$

so it is invariant. $\epsilon_{i_1 i_2}$ transforms as

$$[T^a\epsilon]_{i_1 i_2} = [T^a]^k_{i_1}\epsilon_{k i_2} + [T^a]^k_{i_2}\epsilon_{i_1 k} \quad (12.30)$$

since ϵ is anti-symmetric only independent component is ϵ_{12}

$$[T^a\epsilon]_{12} = [T^a]_1^1\epsilon_{12} + [T^a]_2^2\epsilon_{12} = \epsilon_{12}tr[T^a] = 0 \tag{12.31}$$

so ϵ is invariant.

You can raise indices with ϵ^{ij} so sufficient to consider only upper index tenser in $SU(2)$. For a tenser $\tau^{i_1 i_2 \cdots i_n}$ applying $\epsilon_{i_r i_s}$ on the antisymmetric components give invariant subspaces. Hence totally symmetric requirements are irreducible. Dimension of $\rho^{j_1\cdots j_n}$ (with $i_1\cdots i_n$ symmetrized) can be found as follows: i_k runs over $1, 2$. So linearly independent components of ρ are given by partitioning the set $i_1\cdots i_n$ as $111\cdots 1/222\cdots 2$. The number of ways of doing this is the number of ways you can put one partition among n boxes, i.e., $\begin{pmatrix} n+1 \\ 1 \end{pmatrix} = n+1$ Note that this is the dimension of spin$-\frac{n}{2}$ representation.

From the transformation law L gave above we see that

$$[T_a\rho]^{j_1\cdots j_n} = \sum_{l=1}^{n}[T^a]_k^{j_l}\rho^{j_1\cdots j_{l-1}kj_{l+1}\cdots j_n} \tag{12.32}$$

since $T_3 = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and

$$\rho^{i_1\cdots i_) n} = \rho^{i_1}\otimes 1\otimes\cdots\otimes 1 + 1\otimes\rho^{i_2}\otimes\cdots\otimes 1 + \cdots + 1\otimes\cdots\otimes 1\otimes\rho^{i_n} \tag{12.33}$$

where each covariant vector is a spin$-\frac{1}{2}$ representation, T_3 reads the total S_z (z components of the spin) in the representation $\rho^{i_1\cdots i_n}$. This is in the range $(\frac{n}{2}, -\frac{n}{2})$ so $\rho^{i_1\cdots i_n}$ is indeed a spin$-\frac{n}{2}$ representation and each state in this representation is labeled by the eigenvalue of T_3. Bells are ringing.

(d) Tenser products of representation of the group is $R_1 \otimes R_2$. Since group elements are obtained by erspenentrating the algebra $G = e^T$, tenser products of the representation of the algebra are of the form $r_1 \otimes 1_2 + 1_1 \otimes r_2$. This obviously satisfy the same commutation relations.

Let me only show the evaluation of $C_2(\rho)$ in the most non-trivial example, $C_2(27)$ in $SU(3)$. Consider the Clebsh-Gordon decomposition of a product representation:

$$\rho_1 \otimes \rho_2 = \sum_i \rho_i \tag{12.34}$$

The way T^a acts on $\rho_1 \otimes \rho_2$ is given above

$$T^a_{\rho_1\otimes\rho_2} = T^a_{\rho_1}\otimes 1_{\rho_2} + 1_{\rho_1}\otimes T^a_{\rho_1} \tag{12.35}$$

So

$$tr(T^a_{\rho_1\otimes\rho_2}T^a_{\rho_1\otimes\rho_2}) = (C_2(\rho_1)+C_2(\rho_2))d\rho_1 d\rho_2 \tag{12.36}$$

On the other hand,

$$\begin{aligned} T^a_{\rho_1\otimes\rho_2} &= \sum_i T^a_{\rho_i} &(12.37)\\ tr(T^a_{\rho_1\otimes\rho_2}T^a_{\rho_1\otimes\rho_2}) &= tr(\sum_i T^a_{\rho_i}\sum_j T^a_{\rho_j}) &(12.38)\\ &= \sum_i tr(T^a_{\rho_j}T^a_{\rho_j}) &(12.39)\\ &= \sum_i C_2(\rho_i)d_{\rho_i} &(12.40)\end{aligned}$$

Then,

$$(C_2(\rho_1)+C_2(\rho_2))d\rho_1 d\rho_2 = \sum_i C_2(\rho_i)d_{\rho_i} \tag{12.41}$$

27 occurs in the product of two 8's:

$$8\times 8 = 27 + 10 + \bar{1}0 + 8 + 8 + 1 \tag{12.42}$$

You should have found that $C_2(8) = 3$, $C_2(10) = 6$. Plug these in:

$$\begin{aligned} (3+3)\cdot 8\cdot 8 &= C_2(27)\cdot 27 + 2\cdot 6\cdot 10 + 2\cdot 3\cdot 8 + 0 &(12.43)\\ 8\cdot 8\cdot 6 &= 27C_2(27) + 168 &(12.44)\\ C_2(27) &= \frac{216}{27} &(12.45)\\ &= 8 &(12.46)\end{aligned}$$