8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

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Lecture 7

2: GENERAL ASPECTS

Now that we have studied almost all known quantum field theories, we return to studying physical observables in these theories. We will start with the simplest object:

$$\langle 0| O(x)O(y) |0\rangle , \tag{1}$$

that is, the vacuum two-point function. It is also one of the most important objects. We will first describe its general structure based on symmetries, unitarity and analyticity, without the specifics of any particular theory. We will then consider an example in a specific theory.

2.1: FIELD AND MASS RENORMALIZATIONS

For illustration, consider a scalar field theory, with Lagrangian $\mathscr{L}(\phi)$ which is Lorentz invariant and translation invariant. We examine

$$\begin{aligned} G_F(x,y) &= \langle 0| T(\phi(x)\phi(y)) |0\rangle \text{ , the Feynman function,} \\ G_+(x,y) &= \langle 0| \phi(x)\phi(y) |0\rangle \text{ , the Wightman function, and} \\ G_R(x,y) &= \Theta(x^0-y^0) \langle 0| \phi(x)\phi(y) |0\rangle \text{ , the retarded Green's function.} \end{aligned}$$

Here, $\Theta(x^0)$ is the heaviside function. Translation invariance implies that $G_F(x,y) = G_F(x-y)$, and Lorentz invariance implies that $G_F(x-y) = G_F((x-y)^2)$. In momentum space, we have that

$$G_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip.(x-y)} G_F(p), \tag{2}$$

and by Lorentz invariance, $G_F(p)$ depends only on p^2. Recall, in the free theory,

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2, \tag{3}$$

and $G_+^{(0)}(p;m_0^2) = 2\pi\Theta(p^0)\delta(p^2+m_0^2)$. The delta function constrains the support of the Green's function to the mass-shell, and the heaviside function constrains the support to the positive-energy sheet.

$$G_F(p;m_0^2) = \frac{-i}{p^2+m_0^2-i\epsilon}. \tag{4}$$

Now, for a general interacting theory,

$$G_+(x,y) = \sum_n \langle 0| \phi(x) |n\rangle \langle n| \phi(y) |0\rangle , \tag{5}$$

where we have just inserted a complete set of states, and where the summation heuristically also represents an integration over momentum. Now,

$$\begin{aligned} \langle 0| \phi(x) |n\rangle &= \langle 0| e^{-i\hat{p}.x}\phi(0)e^{i\hat{p}.x} |n\rangle \\ &= e^{ip_n.x} \langle 0| \phi(0) |n\rangle , \end{aligned}$$

and so,

$$G_+(x-y) = \sum_n e^{ip_n.(x-y)} |\langle 0| \phi(0) |n\rangle|^2 , \tag{6}$$

or, more explicitly,

$$G_+(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip.(x-y)} (2\pi)^4 \sum_n \delta^{(4)}(p-p_n) \left|\langle 0|\,\phi(0)\,|n\rangle\right|^2 . \tag{7}$$

We define $(2\pi)\Theta(p^0)\rho(-p^2) = (2\pi)^4 \sum_n \delta^{(4)}(p-p_n) \left|\langle 0|\,\phi(0)\,|n\rangle\right|^2$, a Lorentz invariant scalar function of p, with $\rho(-p^2) = 0$ for $p^2 > 0$, as all $|n\rangle$ should have $E_n = p_n^0 > 0$, and so $p_n^2 < 0$. We thus find that

$$\begin{aligned} G_+(p) =& 2\pi\Theta(p^0)\rho(-p^2) \\ =& \int_0^\infty d\mu^2\, 2\pi\delta(p^2+\mu^2)\Theta(p^0)\rho(\mu^2), \end{aligned}$$

and so, for an interacting theory, we obtain the general result:

$$\begin{aligned} G_+(p) &= \int_0^\infty d\mu^2\, \rho(\mu^2) G_+^{(0)}(p;\mu^2) \\ G_+(x-y) &= \int_0^\infty d\mu^2\, \rho(\mu^2) G_+^{(0)}(x-y;\mu^2) \end{aligned}$$

where $\rho(\mu^2)$ is the spectral function. Now we consider the structure of $\rho(-p^2)$. We have that $(2\pi)\Theta(p^0)\rho(-p^2) = (2\pi)^4 \sum_n \delta^{(4)}(p-p_n) \left|\langle 0|\,\phi(0)\,|n\rangle\right|^2$, where $\sum_n$ is a sum over all physical states: that is, a sum over single-particle states and multi-particle states. Firstly,

$$\begin{aligned} \sum_{\text{single-particle states}} &= \sum_i \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_k} \\ &= \sum_i \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2+m_i^2), \end{aligned}$$

where i indexes the particle species, and $\omega_k = \sqrt{m_i^2 + \vec{k}^2}$. Other than the m_i^2, this sum is completely determined by Lorentz symmetry. Now, we consider the multi-particle states. For a given $\vec{k}$, a continuum of ω are allowed. For example, for a two-particle state at $\vec{k} = 0$,

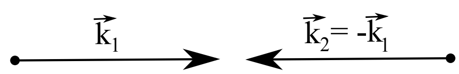

Figure 1: A two particle-state with zero total momentum.

$$\omega = \sqrt{k_1^2 + m_1^2} + \sqrt{k_2^2 + m_2^2}, \tag{8}$$

ω forms a continuum starting at $m_1 + m_2$, and so, k^2 starts at $-(m_1+m_2)^2$. We thus find

$$(2\pi)\Theta(p^0)\rho(-p^2) = \sum_i \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2+m_i^2)\Theta(k^0)(2\pi)^4\delta^{(4)}(p-k) \left|\langle 0|\,\phi(0)\,|k,i\rangle\right|^2 + 2\pi\Theta(p^0)\sigma(-p^2) \tag{9}$$

where the second term is the contribution from multi-particle states. Hence, we have that

$$\rho(-p^2) = \sigma(-p^2) + \sum_i \delta(p^2+m_i^2) Z_i \tag{10}$$

with $Z_i = \left|\langle 0|\,\phi(0)\,|k,i\rangle\right|^2$ a number which is independent of k, since $k^2 = -m_i^2$. $\sigma(-p^2)$ is non-zero for $-p^2 \geq 4m_1^2$, where m_1 is the smallest single-particle mass. So, for the Feynman function, we have

$$G_F(p) = \sum_j \frac{-iZ_j}{p^2 + m_j^2 - i\epsilon} + \int_{4m_1^2}^\infty d\mu^2\, \sigma(\mu^2) \frac{-i}{p^2+\mu^2-i\epsilon}. \tag{11}$$

It is convenient to introduce a function defined for a general complex parameter s,

$$\Gamma(s) \equiv \sum_j \frac{-iZ_j}{s - m_j^2} + \int_{4m_1^2}^\infty d\mu^2\, \sigma(\mu^2) \frac{-i}{s-\mu^2}. \tag{12}$$

Then $G_F(p^2) = \Gamma(s = -p^2 + i\epsilon)$, and so, we obtain the Feynman function by approaching the real $s-$axis from above. We observe two features of $\Gamma(s)$:

1. $\Gamma(s)$ has poles at single-particle mass-squared values: $s = m_j^2$.

2. There is a branch cut beginning at $4m_1^2$ with a discontinuity $\Gamma(r + i\epsilon) - \Gamma(r - i\epsilon) = 2\pi\sigma(r)$.

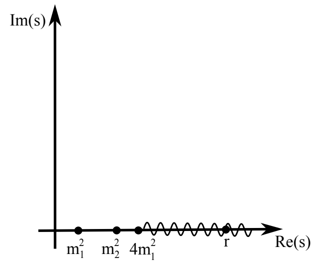

Figure 2: The function $\Gamma(s)$ plotted on the complex plane.

Now, consider

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 + \mathscr{L}_I(\phi, \lambda), \tag{13}$$

and suppose that as $\lambda \longrightarrow 0$, $\mathscr{L} \longrightarrow 0$. At $\lambda = 0$, $G_F^{(0)}(p) = \frac{-i}{p^2+m_0^2-i\epsilon}$. If ϕ does not have any bound states, then, for $\lambda \neq 0$, we have

$$G_F(p) = \frac{-iZ}{p^2 + m^2 - i\epsilon} + \int_{4m^2}^{\infty} d\mu^2\, \sigma(\mu^2) \frac{i}{p^2 + \mu^2 - i\epsilon}. \tag{14}$$

In general, we note the following points:

1. $m^2 \neq m_0^2$: we have a mass renormalization. m_0^2, the bare mass which appears in the Lagrangian, is not necessarily m^2, the physical mass which appears in the propagator.

2. $Z \neq 1$: we have a field-strength renormalization. $Z = |\langle 0|\, \phi(0)\, |k\rangle|^2$. These first two points are generic to interactions, and have nothing to do with ultraviolet divergences.

3. $\phi(0)\, |0\rangle$ also generates multiple-particle states.

4. We define $\phi_{phys} \equiv \sqrt{Z}\phi$, meaning $|\langle 0|\, \phi_{phys}(0)\, |k\rangle|^2 = 1$.

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