8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 6

1.5: BRST SYMMETRY, PHYSICAL STATES AND UNITARITY

1.5.1: Becchi-Rouet-Stora-Tyutin (BRST) Symmetry

From the last lecture, we have

$$Z = \int \mathfrak{D}A_\mu^a \mathfrak{D}C_a D\bar{C}_a \, e^{iS_{eff}[A_\mu^a, C, \bar{C}]}, \tag{1}$$

with

$$S_{eff}\left[A, C, \bar{C}\right] = S_0\left[A\right] - \frac{1}{2\xi}\int d^4x \, f_a^2(A) + \int d^4x d^4y \, \bar{C}_a(x) \left[\frac{\delta f_a(A_\Lambda(x))}{\delta \Lambda_b(y)}\bigg|_{\Lambda=0} \right] C_b(y), \tag{2}$$

where $f_a(A)$ is the gauge-fixing function and $S_0\left[A\right] = \frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a}$ is the pure Yang-Mills action. $S_0\left[A\right]$ is invariant under the gauge transformation

$$A_\mu^a \longrightarrow A_\mu^a + D_\mu \Lambda^a \tag{3}$$

where $D_\mu \Lambda^a = \partial_\mu \Lambda^a + f^{abc} A_\mu^b \Lambda^c$. We note that in (1) we integrate over all $A_\mu^a(x)$, including the unphysical configurations, but by construction Z should only receive contributions from the physical $A_\mu^a(x)$. We also note that $Z = \langle 0, +\infty | \, 0, -\infty\rangle$. $S_{eff}\left[A, C, \bar{C}\right]$ no longer has gauge symmetries, but it has a hidden global fermionic symmetry, the BRST symmetry, which is, in fact, a remnant of the gauge symmetry. To see this, it is convenient to introduce an auxillary field $h_a(x)$:

$$Z = \int \mathfrak{D}A_\mu^a \mathfrak{D}h_a \mathfrak{D}C_a D\bar{C}_a \, e^{iS_{eff}[A_\mu^a, C, \bar{C}, h]} \tag{4}$$

with

$$S_{eff}\left[A, C, \bar{C}, h\right] = S_0\left[A\right] + \frac{\xi}{2}\int d^4x \, h_a^2 + \int d^4x \, h_a(x) f_a(x) + \mathscr{L}_{gh}. \tag{5}$$

Now, consider the following (BRST) transformations:

$$\delta_B A_\mu^a = \eta (D_\mu C)^a \equiv \eta s(A_\mu^a) \tag{6}$$

$$\delta_B \bar{C}^a = -\eta h^a \equiv \eta s(\bar{C}^a) \tag{7}$$

$$\delta_B C^a = -\frac{1}{2} g \eta f^{abc} C^b C^c \equiv \eta s(C^a) \tag{8}$$

$$\delta_B h^a = 0 \equiv \eta s(h^a) \tag{9}$$

with η an anticommuting constant parameter. Then, in general,

$$\delta_B \phi \equiv \eta s(\phi), \quad \phi = A_\mu^a, C^a, \bar{C}^a, h^a. \tag{10}$$

$s(\phi)$ takes ϕ to a field of opposite 'fermionic parity'. We note some of the important properties of s :

i.

$$s(\phi_1 \phi_2) = s(\phi_1)\phi_2 \pm \phi_1 s(\phi_2), \tag{11}$$

where the $+$ sign is for ϕ_1 bosonic, and the $-$ sign is for ϕ_1 fermionic.

ii.

$$s^2(\phi) = 0. \tag{12}$$

For example, $s^2(\bar{C}^a) = 0$ and $s^2(C^a) = 0$, which follows from the Jacobi identity.

iii. From (i) and (ii), we have that

$$s^2(F(\phi)) = 0. \tag{13}$$

iv. $s(A^a_\mu)$ is the same as the infinitesimal gauge transformation of A^a_μ with Λ^a replaced by C^a.

Based on the above properties, we will now prove that $\delta_B S = 0$.

We first show that

$$S = S_0 + \int d^4x \, s(F(x)) \tag{14}$$

with $F(x) = -\bar{C}_a f_a - \frac{\xi}{2}\bar{C}_a h_a$, so that

$$s(F(x)) = h_a f_a + \bar{C}_a s(f_a(A_\mu)) + \frac{\xi}{2} h_a^2. \tag{15}$$

This can be established by showing that

$$\int d^4x \, \bar{C}_a(x) s(f_a(A_\mu(x))) = \int d^4y d^4x \, \bar{C}_a(x) \left[\left. \frac{\delta f_a(A_\Lambda(x)}{\delta \Lambda_b(y)} \right|_{\Lambda=0} \right] C_b(y), \tag{16}$$

which is left as an exercise to the reader. Then, we have that

$$\delta_B S = \delta_B S_0 + \eta \int d^4x \, s^2(F(x)), \tag{17}$$

and these terms are separately zero by the properties (iii) and (iv) shown above.

□

The BRST symmetry implies the existence of a conserved fermionic charge Q_B.

$$\delta_B \phi = i \left[\eta Q_B, \phi \right] = \eta s(\phi) \tag{18}$$

or, equivalently,

$$\begin{aligned} s(\phi) =& i \left[Q_B, \phi \right]_\pm \\ =& \begin{cases} i \left[Q_B, \phi \right], & \phi \text{ bosonic}, \\ i \left\{ Q_B, \phi \right\}, & \phi \text{ fermionic}. \end{cases} \end{aligned}$$

Since $s^2(\phi) = 0$, we have that

$$\left[Q_B, \left[Q_B, \phi \right]_\pm \right]_\mp = 0. \tag{19}$$

That is, $\left[Q_B^2, \phi \right] = 0$ for any ϕ, and hence,

$$Q_B^2 = 0. \tag{20}$$

We can also define a ghost number, which is conserved:

$$\text{gh}\left[C \right] = 1, \; \text{gh}[\bar{C}] = -1, \; \text{gh}\left[Q_B \right] = 1, \; \text{gh}[\tilde{\phi}] = 0 \tag{21}$$

for any other field $\tilde{\phi}$.

1.5.2: Physical States and Unitarity

Physical states should be independent of the gauge choice. $Z = \langle 0, +\infty | \, 0, -\infty \rangle$ is so by construction, as it should be independent of $f_a(A)$. We now consider more general observables. More generally, we should have that

$$0 = \delta_g \langle f | \, i \rangle = i \langle f | \, \delta_g S \, | i \rangle \tag{22}$$

where δ_g represents the change under the variation of the gauge-fixing condition $f_a(A)$. Note from (14) we have that

$$\begin{aligned}\delta_g S &= \int d^4x\, s(\delta_g F(x)) \\ &= -\int d^4x\, s(\bar{C}_a \delta f_a) \\ &= -i\int d^4x\, \left\{Q_B, \bar{C}_a \delta f_a(A)\right\},\end{aligned}$$

and so it must be true that

$$\int d^4x \left\langle f\right| \left\{Q_B, \bar{C}_a \delta f_a(A)\right\} \left|i\right\rangle = 0 \tag{23}$$

for arbitrary $\delta f_a(A)$ for a physical observable, and so

$$Q_B \left|i\right\rangle = Q_B \left|f\right\rangle = 0. \tag{24}$$

That is, a physical state $|\psi\rangle$ should satisfy

$$Q_B \left|\psi\right\rangle = 0. \tag{25}$$

Similarly, by considering

$$\delta_g \left\langle f\right| O_1 \dots O_n \left|i\right\rangle = 0, \tag{26}$$

we find that

$$[Q_B, O] = 0 \tag{27}$$

and so O should be gauge invariant (if it does not contain ghost fields). Note that any state of the form

$$\left|\psi\right\rangle = Q_B \left|\dots\right\rangle \tag{28}$$

satisfies $Q_B|\psi\rangle = 0$, but that in this case, $\langle\chi|\,\psi\rangle = 0$ for any physical state $|\chi\rangle$. Such a state $|\psi\rangle$ is called a null state. All physical observables involving a null state vanish. If $|\psi'\rangle$, $|\psi\rangle$ satisfying (25) are related by

$$\left|\psi'\right\rangle = \left|\psi\right\rangle + Q_B \left|\dots\right\rangle, \tag{29}$$

they will have the same inner product with all physical states, and thus are equivalent. We introduce

$$\begin{aligned}\mathscr{H}_{closed} &= \left\{\left|\psi\right\rangle : Q_B\left|\psi\right\rangle = 0\right\}, \\ \mathscr{H}_{exact} &= \left\{\left|\psi\right\rangle : \left|\psi\right\rangle = Q_B\left|\dots\right\rangle\right\}, \\ \mathscr{H}_{phys} &= \frac{\mathscr{H}_{closed}}{\mathscr{H}_{exact}}.\end{aligned}$$

That is, $\mathscr{H}_{phys}$ is the cohomology of Q_B. In summary:

1. Defining $\mathscr{H}_{big}$ to be the Fock space composed from $A_\mu, C, \bar{C}$, we have that

$$\mathscr{H}_{phys} \subset \mathscr{H}_{closed} \subset \mathscr{H}_{big}. \tag{30}$$

2. By restricting to $\mathscr{H}_{phys}$ and gauge invariant O, $\langle f|\, O_1 \dots O_n \,|i\rangle$ does not depend on the gauge choice.

3. Our path integral construction guarantees that only physical states contribute in the intermediate state.

Example 1: Quantum electrodynamics in Feynman gauge ($\xi = 1$)

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \partial_\mu C\partial^\mu \bar{C} \\ &= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \partial_\mu C\partial^\mu \bar{C}\end{aligned}$$

Under the BRST transformation:

$$\begin{aligned} \delta_B A_\mu =& \eta \partial_\mu C \\ \delta_B \bar{C} =& -\eta \partial_\mu A^\mu \\ \delta_B C =& 0, \end{aligned}$$

and so

$$\begin{aligned} [Q_B, A_\mu] =& -i\partial_\mu C \\ \left[Q_B, \bar{C}\right] =& i\partial_\mu A^\mu \\ [Q_B, C] =& 0. \end{aligned}$$

It is left as an exercise for the reader to find the explicit form for Q_B. Now, we set $\mathscr{H}_{big}$ to be the set of states formed by acting with creation operators for $A_\mu, \bar{C}, C$ on the ground state $|0\rangle$. Imposing $Q_B |\psi\rangle = 0$ gives us $\mathscr{H}_{closed}$ and $\mathscr{H}_{phys}$. For illustration, consider the one-particle state:

$$\mathscr{H}_{big} = \{|e_\mu, p\rangle, |c, p\rangle, |\bar{c}, p\rangle\}. \tag{31}$$

Then, $Q_B |e, p\rangle = Q_B e \cdot A |0\rangle = -e \cdot p |c, p\rangle$ from $[Q_B, A_\mu] = -i\partial_\mu C$, and we obtain the physical state condition:

$$e.p = 0 \tag{32}$$

For $e.p \neq 0$, we get$|c, p\rangle$ null states. $Q_B |\bar{c}, p\rangle \propto p_\mu A^\mu(p) |0\rangle \neq 0$, and so the $|\bar{c}, p\rangle$ are non-physical states, and $p_\mu A^\mu(p) |0\rangle = |e = p, p\rangle$ is a null state. So, we have that

$$\mathscr{H}_{phys} = \{|e, p\rangle : e.p = 0, e^\mu \sim e^\mu + p^\mu\}. \tag{33}$$

Take $p^\mu = \left(p^0, 0, 0, p^3\right)$, $p^2 = 0$. Then, $e.p = 0$ implies that $e^\mu = \left(p^0, e_1, e_2, p^3\right)$, and $e \sim e + p$ implies that $e^\mu = (0, e_1, e_2, 0)$, and so, only transverse components of A_μ generate physical states.

Remarks:

1. While $A_0 |0\rangle$ creates negative-norm states, they do not lie in the physical state space (giving a positive-definite norm on the physical state space).

2. Ghosts $C, \bar{C}$ make sure that these negative norm states do not contribute in intermediate steps.

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