8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 3

We begin with some comments concerning gauge-symmetric theories:

1. A $U(1)$ local symmetry leads to the field $A_\mu(x)$, mediating interactions between the charge fields $\psi_i(x)$.

2. No mass term is allowed for A_μ or the gauge symmetry is broken.

3. It is possible to construct theories using other gauge invariant terms, for example

$$\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho},\ (F_{\mu\nu}F^{\mu\nu})^2,\ \bar{\psi}F_{\mu\nu}\gamma^\mu\gamma^\nu\psi. \tag{1}$$

All of these terms can be written as a total derivative, or are non-renormalizable.

4. Some of the terminology and constructs associated with gauge theories includes:

$$\begin{aligned} U(y,x) &: \text{Parallel transport},\\ A_\mu(x) &: \text{Connection},\\ F_{\mu\nu}(x) &: \text{Curvature},\\ D_\mu(x) &: \text{Covariant derivative.} \end{aligned}$$

These objects form the mathematical framework of the **fibre bundle**.

5. Gauge symmetry is not really a symmetry. The phase of ψ (and part of A_μ) does not carry physical information; there is a redundancy in our variables.

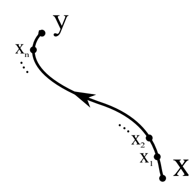

Figure 1: Dividing the path γ into infinitesimal segments.

6. Finally, we consider the explicit form of the finite parallel transport, $U(y,x)$: choose a path γ from $x \longrightarrow y$. We split the path into infinitesimal segments:

$$U_\gamma(y,x) = U(y,x_n)U(x_n,x_{n-1})\ldots U(x_1,x), \tag{2}$$

where, from our result from the previous lecture for the infinitesimal parallel transport,

$$U(x_1,x) \approx \exp\left[ieA_\mu(x_1-x)^\mu\right], \tag{3}$$

and hence

$$U_\gamma(y,x) = \exp\left[ie\int_\gamma A_\mu(x)dx^\mu\right]. \tag{4}$$

Note that $U_\gamma(y,x)$ is not necessarily path-independent: in general, $U_{\gamma_1}(y,x) \neq U_{\gamma_2}(y,x)$. Let

$$U_\Gamma(x,x) = U_{-\gamma_2}(y,x)U_{\gamma_1}(y,x) \tag{5}$$

be the parallel transport associated with the closed loop shown in figure 2. By (4),

$$U_\Gamma(x,x) = \exp\left[ie\oint_\Gamma A_\mu dx^\mu\right],\ (\Gamma = \gamma_1 - \gamma_2). \tag{6}$$

Using Stoke's theorem, we will see in the problem set that

$$U_\Gamma(x,x) = \exp\left[ie\int_\Sigma F_{\mu\nu}dx^\mu dx^\nu\right]. \tag{7}$$

$U_\Gamma(x)$ is the Wilson loop, and it is associated with phenomena such as the Aharonov-Bohm effect, and Berry's phase.

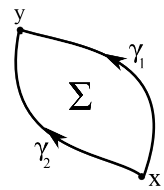

Figure 2: Paths γ_1 and γ_2 from x to y, and the enclosed area Σ.

1.3: NON-ABELIAN GENERALIZATIONS: YANG-MILLS THEORY

Now consider $\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$ and search for a theory invariant under

$$\Psi(x) \longrightarrow \Psi'(x) = V(x)\Psi(x), \tag{8}$$

or, with indices restored,

$$\Psi_i(x) \longrightarrow \Psi'_i(x) = V_i^{\ j}(x)\Psi_j(x). \tag{9}$$

Here, $V(x)$ is an $n \times n$ unitary matrix of unit determinant, that is, $V(x) \in SU(n)$. Let us construct the non-Abelian generalizations of the objects we studied in the Abelian case.

A. Covariant derivative:

Introduce $U(y,x) \in SU(n)$, transforming as

$$U(y,x) \longrightarrow V(y)U(y,x)V^\dagger(x). \tag{10}$$

Again, for $y^\mu = x^\mu + \epsilon n^\mu$, taking the limit $\epsilon \longrightarrow 0$, we expand $U(x+\epsilon n, x)$:

$$U(x+\epsilon n, x) = 1 + ig\epsilon n^\mu A_\mu(x) + \ldots, \tag{11}$$

where g is a constant, and $A_\mu(x)$ is an $n \times n$ matrix. As $U(y,x) \in SU(n)$, $A_\mu(x)$ is also necessarily traceless and hermitian; $A_\mu(x) = A_\mu^\dagger(x)$. Inserting this expansion into the transformation law (10), we obtain the gauge transformation law for the connection:

$$A_\mu(x) \longrightarrow V(x)A_\mu(x)V^\dagger(x) - \frac{i}{g}(\partial_\mu V(x))V^\dagger(x). \tag{12}$$

As before, we define the covariant derivative by

$$n^\mu D_\mu \Psi \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\Psi(x+\epsilon n) - U(x+\epsilon n, x)\Psi(x)\right]. \tag{13}$$

Hence, we have, with indices written explicitly,

$$(D_\mu \Psi)_i = \partial_\mu \Psi_i - ig(A_\mu)_i^{\ j}\Psi_j. \tag{14}$$

From (8) and (12), we have that, under a gauge transformation, $D_\mu\Psi(x) \longrightarrow V(x)(D_\mu\Psi)(x)$, and so, the Lagrangian $\mathscr{L} = -i\overline{\Psi}(\gamma^\mu D_\mu - m)\Psi$ is invariant.

B. Kinetic term for A_μ :

We note that under a gauge tranformation,

$$[D_\mu, D_\nu]\Psi \longrightarrow V[D_\mu, D_\nu]\Psi, \tag{15}$$

and that

$$\begin{aligned} [D_\mu, D_\nu] &= [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] \\ &= -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]) \\ &\equiv -igF_{\mu\nu}, \end{aligned} \tag{16}$$

with $F_{\mu\nu}$ an $n\times n$ matrix satisfying $F_{\mu\nu}=F^{\dagger}_{\mu\nu}$ and $\mathrm{Tr}F_{\mu\nu}=0$. Under a gauge transformation, from (8), we have that

$$F_{\mu\nu}\Psi \longrightarrow F'_{\mu\nu}\Psi' = VF_{\mu\nu}\Psi, \tag{17}$$

and, as $\Psi' = V\Psi$, we have that

$$F'_{\mu\nu}(x) = V(x)F_{\mu\nu}(x)V^{\dagger}(x), \tag{18}$$

and so $F_{\mu\nu}$ is gauge covariant, as can be checked directly from (12). Hence, $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ is invariant under gauge transformations.

C. The Lagrangian:

We can now write down an invariant $\mathscr{L}$:

$$\mathscr{L} = -\frac{c}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - i\bar{\Psi}(\gamma^{\mu}D_{\mu}-m)\Psi, \tag{19}$$

with $\Psi = \begin{pmatrix}\psi_1\\ \vdots \\ \psi_n\end{pmatrix}$, $D_\mu = \partial_\mu - igA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]$, and $A^{\dagger}_{\mu} = A_\mu$, $\mathrm{Tr}A_\mu = 0$. Explicitly, this Lagrangian is invariant under the local gauge transformation:

$$\begin{aligned}\Psi(x) &\longrightarrow \Psi'(x) = V(x)\Psi(x),\\ A_\mu(x) &\longrightarrow A'_\mu(x) = V(x)A_\mu(x)V^{\dagger}(x) - \frac{i}{g}(\partial_\mu V(x))V^{\dagger}(x),\\ F_{\mu\nu}(x) &\longrightarrow F'_{\mu\nu}(x) = V(x)F_{\mu\nu}(x)V^{\dagger}(x),\end{aligned}$$

where $V(x) = \exp[i\Lambda^a(x)T_a]$, and T_a are the generators of the Lie algebra, satisfying $[T_a, T_b] = if_{abc}T_c$.

Remarks:

1. A_μ is massless; introducing a mass term breaks gauge invariance.

2. It is convenient to expand A_μ as $A_\mu = A^a_\mu T_a$, with $a = 1, 2, \cdots n^2-1$. Here, A_μ is an $n\times n$ matrix, and A^a_μ are n^2-1 ordinary functions of x. Similarly, $F_{\mu\nu} = F^a_{\mu\nu}T_a$. We have that

$$F^a_{\mu\nu}T_a = \partial_\mu A^a_\nu T_a - \partial_\nu A^a_\mu T_a - ig\left[A^b_\mu T_b, A^c_\nu T_c\right], \tag{20}$$

and so, explicitly,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{abc}A^b_\mu A^c_\nu. \tag{21}$$

3. There is another gauge invariant term which can be constructed out of F at the quadratic level:

$$\epsilon_{\mu\nu\lambda\rho}\mathrm{Tr}(F^{\mu\nu}F^{\lambda\rho}). \tag{22}$$

However, this term breaks CP-symmetry, and is also a total derivative. Nevertheless, this term is important at a non-perturbative level, as we will see in 8.325.

4. Non-Abelian gauge fields are associated with fibre bundles.

1.3.2: The Wilson loop

$$\begin{aligned}U_\gamma(y,x) &= \lim_{n\to\infty} U(y,x_n)U(x_n,x_{n-1})\dots U(x_1,x) \\ &= \lim_{\Delta x_j\to 0}\prod_{j=0}^{n}(1+igA_\mu(x_j)\Delta x^\mu_j),\end{aligned} \tag{23}$$

with $\Delta x_j^\mu = x_{j+1}^\mu - x_j^\mu$, $x_0 = x$, $x_{n+1} = y$. Note that the ordering is important in (23), as A_μ is a matrix, and so $[A_\mu(x_j), A_\nu(x_k)] \neq 0$ in general.

$$U_\gamma(y,x) = 1 + ig\sum_{j=0}^{n} A_\mu(x_j)\Delta x_j^\mu + (ig)^2 \sum_{j=0}^{n}\sum_{k=0}^{j-1} A_\mu(x_j)\Delta x_j^\mu A_v(x_k)\Delta x_k^v + \ldots + . \tag{24}$$

Now, we introduce $x^\mu(s)$ to parameterise γ:

$$x^\mu(0) = 0,\; x^\mu(1) = y^\mu,\; s \in [0,1]\,. \tag{25}$$

Then

$$\begin{aligned} U_\gamma(y,x) =& 1 + ig\int_0^1 ds_1 A_\mu(x(s_1))\frac{dx^\mu}{ds_1} + (ig)^2\int_0^1 ds_1 \int_0^{s_1} ds_2 A_\mu(x(s_1))\frac{dx^\mu}{ds_1} A_v(x(s_2))\frac{dx^v}{ds_2} + \ldots + & (26)\\ =& \sum_{n=0}^{\infty}(ig)^n \int_0^1 ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n A_{\mu_1}(x(s_1))\frac{dx^{\mu_1}}{ds_1}\ldots A_{\mu_n}(x(s_n))\frac{dx^{\mu_n}}{ds_n} & (27)\\ \equiv& P\exp\left[ig\int_0^1 ds\frac{dx^\mu}{ds}A_\mu(x(s))\right] & (28)\\ =& P\exp\left[ig\int_\gamma A_\mu(x)dx^\mu\right]. & (29) \end{aligned}$$

By construction, under a gauge transformation,

$$U_\gamma(y,x) \longrightarrow V(y)U_\gamma(y,x)V^\dagger(x). \tag{30}$$

To prove this directly using (12) is slightly non-trivial. As in the Abelian case, in general $U_{\gamma_1}(y,x) \neq U_{\gamma_2}(y,x)$. For a closed loop Γ, $U_\Gamma(x,x)$ is nontrivial. The non-Abelian generalisation of Stokes' theorem can be used to relate the parallel transport around the loop to the flux passing through the loop. For an infinitesimal loop,

$$U_\Gamma(x,x)\Psi - \Psi = \frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}\Psi, \tag{31}$$

where $\sigma^{\mu\nu}$ is the area element encircled by the loop. Under a gauge transformation,

$$U_\Gamma(x,x) \longrightarrow V(x)U_\Gamma(x,x)V^\dagger(x), \tag{32}$$

and hence,

$$W_\Gamma(x) = \mathrm{Tr}(U_\Gamma(x,x)) = \mathrm{Tr}(P\exp ig\oint_\Gamma A_\mu dx^\mu) \tag{33}$$

is gauge invariant. This is the non-Abelian Wilson loop. It is a very important object.

8.324 Relativistic Quantum Field Theory II
Fall 2010