8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2010

Lecture 26

5.3.3: Beta-Functions of Quantum Electrodynamics

In the case of quantum electrodynamics, we have the Lagrangian

$$\mathscr{L} = -\frac{1}{4}F^B_{\mu\nu}F^{\mu\nu}_B - i\bar{\psi}_B\left(\gamma^\mu\left(\partial_\mu - ie_B A^B_\mu\right) - m_B\right)\psi_B, \tag{1}$$

with $\psi_B = Z_2^{\frac{1}{2}}\psi$, $A^B_\mu = Z_3^{\frac{1}{2}}A_\mu$, $m_B = m + \delta m$, and $e_B = Z_3^{-\frac{1}{2}}e\mu^{\frac{\epsilon}{2}}$ or $\alpha_B = Z_3^{-1}\alpha\mu^\epsilon$, where $\alpha = \frac{e^2}{4\pi}$. From our earlier result that $Z_3 = 1 - \frac{2\alpha}{3\pi}\frac{1}{\epsilon}$, we have

$$\alpha_B = \mu^\epsilon\left[\alpha + \frac{2\alpha^2}{3\pi}\frac{1}{\epsilon} + \ldots\right], \tag{2}$$

and

$$\beta_\alpha = -\frac{2\alpha^2}{3\pi} + \frac{4\alpha^2}{3\pi} = \frac{2\alpha^2}{3\pi} + O(\alpha^3). \tag{3}$$

Hence, the running of $\alpha(\mu)$ is given by

$$\frac{1}{\alpha(\mu)} = \frac{1}{\alpha(\mu_0)} - \frac{2}{3\pi}\log\frac{\mu}{\mu_0}, \tag{4}$$

or, equivalently,

$$\alpha(\mu) = \frac{\alpha_0}{1 - \frac{2\alpha_0}{3\pi}\log\frac{\mu}{\mu_0}}. \tag{5}$$

In quantum electrodynamics, $\alpha(\mu)$ increases as μ is increased, and $\alpha(\mu)$ decreases as μ decreases. In particular,

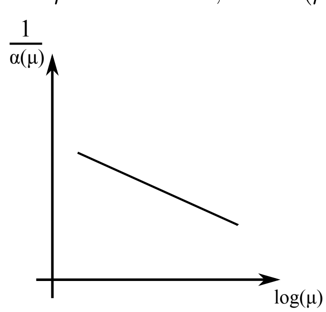
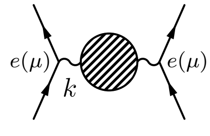


Figure 1: We find a linear relationship between $\alpha^{-1}(\mu)$ and $\log(\mu)$ with a negative gradient for quantum electrodynamics.

the Landau pole at $\alpha(\mu) \longrightarrow \infty$ is given by

$$\mu = \Lambda \equiv \mu_0 e^{\frac{3\pi}{2\alpha_0}}, \tag{6}$$

independently of the choice of μ_0. Quantum electrodynamics becomes strongly coupled near the scale Λ. It is convenient to express $\alpha(\mu)$ in terms of the physical $\alpha_{phys} \approx \frac{1}{137}$ which we measure. Consider



$$\propto \frac{1}{k^2}\frac{e^2(\mu)}{1 - \Pi(k^2)}. \tag{7}$$

At the one-loop level,

$$\Pi(k^2) = -\frac{e^2(\mu)}{\pi^2}\int_0^1 dx\, x(1-x)\left(\frac{1}{\epsilon} - \frac{1}{2}\log\left(\frac{D}{\tilde{\mu}^2}\right)\right) - (Z_3 - 1), \tag{8}$$

where $\tilde{\mu}^2 \equiv \frac{4\pi\mu^2}{e^\gamma}$, $D \equiv m^2 + x(1-x)k^2$ and $m^2 = m_e^2 + O(\alpha)$, where m_e is the physical electron mass. It is convenient to introduce

$$\hat{\alpha}(k) \equiv \frac{\alpha(\mu)}{1-\Pi(k^2)} = \frac{\alpha_B}{1-\Pi_B(k^2)}. \tag{9}$$

Note that this quantity is finite, although the numerator and denominator of the last term are divergent. $\hat{\alpha}(k)$ is an effective k-dependent coupling. In particular,

$$\alpha_e = \hat{\alpha}(k=0) \approx \frac{1}{137}. \tag{10}$$

Now,

$$\Pi_{\overline{MS}}(k^2) = \frac{2\alpha(m)}{\pi}\int_0^1 dx\, x(1-x)\log\frac{D}{\mu^2}, \tag{11}$$

and so

$$\Pi_{\overline{MS}}(k^2=0) = \frac{e^2(\mu)}{6\pi^2}\log\frac{m_e}{\mu} = \frac{2\alpha(\mu)}{3\pi}\log\frac{m_e}{\mu}. \tag{12}$$

We therefore have

$$\alpha_e = \frac{\alpha(\mu)}{1-\frac{2\alpha(\mu)}{3\pi}\log\frac{m_e}{\mu}}. \tag{13}$$

We can compare this to

$$\alpha(\mu') = \frac{\alpha(\mu)}{1-\frac{2\alpha(\mu)}{3\pi}\log\frac{\mu'}{\mu}}, \tag{14}$$

and we find

$$\alpha_e = \alpha(\mu = m_e). \tag{15}$$

Therefore, in the $\overline{\text{MS}}$ scheme,

$$\alpha(\mu) = \frac{\alpha_e}{1-\frac{2\alpha_e}{3\pi}\log\frac{\mu}{m_e}}, \tag{16}$$

and the Landau pole occurs at $\Lambda = m_e e^{\frac{3\pi}{2\alpha_e}} \approx m_e e^{5\times 137}$. We now consider $\hat{\alpha}(k) = \frac{\alpha(\mu_0)}{1-\Pi_{\overline{MS}}(k^2)}$:

1. For $k^2 \gg m_e^2$, $D \sim x(1-x)k^2$, and so

$$\Pi_{\overline{MS}}(k^2) = \frac{\alpha(\mu_0)}{3\pi}\log\frac{k^2}{\mu_0^2} + \dots. \tag{17}$$

Therefore,

$$\hat{\alpha}(k) = \frac{\alpha(\mu_0)}{1-\frac{2\alpha(\mu_0)}{3\pi}\log\frac{k}{\mu_0}} + \dots, \tag{18}$$

and so $\hat{\alpha}(k) \approx \alpha(\mu)$ for $\mu \approx k$. Note that this is scheme-independent: in any scheme for $\mu \gg m_e$, $\alpha(\mu) \approx \hat{\alpha}(\mu)$.

2. When $k^2 \ll m_e^2$, we have $\hat{\alpha}(k) \approx \alpha_e$, but $\alpha(\mu) \longrightarrow 0$ as $\frac{\mu}{m_e} \longrightarrow 0$. For $\mu < m_e$, $\alpha(\mu)$ differs qualitatively from the physical coupling. Physically, m_e becomes important, but that is not tracked by the MS or

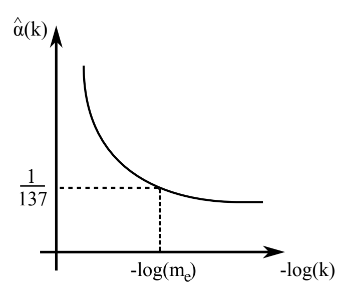


Figure 2: $\hat{\alpha}(k)$ as a function of $\log\frac{1}{k}$. At the scale of $k \sim m_e$, we have $\hat{\alpha}(m_e) \sim \frac{1}{137}$.

$\overline{\text{MS}}$ schemes: $\alpha(\mu)$ is no different for a theory with $m_e = 0$. It is more transparent to understand the behaviour of $\hat{\alpha}(k)$ using the Wilsonian approach. The coupling in the Wilsonian action, by definition, should track $\hat{\alpha}(k)$ closely.

Below the scale of m_e, the electron becomes heavy, and we can then integrate it out, leaving a pure Maxwell theory, in which the coupling constant does not run.

3. For massless quantum electrodynamics, with $m_e = 0$, $\alpha(\mu)$ is qualitatively correct for $\mu \longrightarrow 0$. We then find that $\alpha_{eff}(k) \longrightarrow 0$ as $k \longrightarrow 0$: The theory is marginally irrelevant.

5.3.4: Beta-Function of Quantum Chromodynamics

The Lagrangian of quantum chromodynamics, with an $SU(N_c)$ gauge group and N_f quarks, where $N_c = 3$ and $N_f = 6$, is given by

$$\mathscr{L} = -\frac{1}{4}\text{Tr}F_{\mu\nu}F^{\mu\nu} - i\sum_{j=1}^{N_f}\bar{\psi}_j\left(\gamma^\mu D_\mu - m_j\right)\psi_j, \tag{19}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right],$$

$A_\mu = A_\mu^a t^a$, $F_{\mu\nu} = F_{\mu\nu}^a t^a$, where t^a are the generators of the fundamental representation of $SU(N_c)$. The covariant derivative is given by

$$D_\mu\psi_j = \partial_\mu\psi_j - igA_\mu\psi_j. \tag{20}$$

We redefine $A_\mu \longrightarrow \frac{1}{g}A_\mu$, and so the Lagrangian becomes

$$\mathscr{L} = -\frac{1}{4g^2}\text{Tr}F_{\mu\nu}F^{\mu\nu} - i\sum_{j=1}^{N_f}\bar{\psi}_j\left(\gamma^\mu D_\mu - m_j\right)\psi_j, \tag{21}$$

with

$$D_\mu\psi_j = \partial_\mu\psi_j - iA_\mu\psi_j. \tag{22}$$

We will outline the computation of the $\beta-$function for the pure gauge theory using the language of the Wilsonian approach, in the Euclidean case. Consider that the theory is provided with a cut off Λ, and bare coupling $g = g(\Lambda)$. We now write

$$A_\mu = A_\mu^{(\Lambda')} + \tilde{A}_\mu, \tag{23}$$

where $A_\mu^{(\Lambda')}$ is the part below the scale Λ', and $\tilde{A}_\mu$ is the high-energy part, above the scale Λ'. Then we have

$$S_{YM}\left[A_\mu\right] = S_{YM}\left[A_\mu^{(\Lambda')}\right] + S_1\left[A_\mu^{(\Lambda')}, \tilde{A}_\mu\right], \tag{24}$$

and

$$\begin{aligned}\int \mathfrak{D}A_\mu e^{-S[A_\mu]} &= \int \mathfrak{D}A_\mu^{(\Lambda')} e^{-S\left[A_\mu^{(\Lambda')}\right]}\int \mathfrak{D}\tilde{A}_\mu e^{-S_1\left[A_\mu^{(\Lambda')},\tilde{A}_\mu\right]}\\ &= \int \mathfrak{D}A_\mu^{(\Lambda')} e^{-S\left[A_\mu^{(\Lambda')}\right]-\Delta S\left[A_\mu^{(\Lambda')}\right]}.\end{aligned}$$

We take the derivative expansion of ΔS,

$$\Delta S \approx c\log\frac{\Lambda}{\Lambda'}\int d^4x\, F_{\Lambda'}^2 + \ldots, \tag{25}$$

giving

$$\frac{1}{g^2(\Lambda')} = \frac{1}{g^2(\Lambda)} + c'\log\frac{\Lambda}{\Lambda'}, \tag{26}$$

where c is a pure $g-$independent number. A precise calculation gives $c = -\frac{1}{(4\pi)}\frac{22}{3}N_C$. So, for the $\beta-$function, we find

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N_C - \frac{2}{3}N_F\right). \tag{27}$$

Considering the fermionic sector, we have

$$\begin{aligned}\int \mathfrak{D}\bar{\psi}\mathfrak{D}\psi\, e^{-\int d^dx\,\bar{\psi}(\not{D}_{\Lambda'}-m)\psi+\ldots} &\propto \det\left(\not{D}_{\Lambda'} - m\right) \\ &= e^{\log\det\left(\not{D}_{\Lambda'}-m\right)}.\end{aligned}$$

If we define $\alpha_s = \frac{g^2}{4\pi}$, we find

$$\beta_\alpha = -\frac{\alpha^2}{2\pi}\left(\frac{11}{3}N_C - \frac{2}{3}N_F\right) = -b\alpha^2, \tag{28}$$

as we found in the $g\phi^3$ theory. So, we have

$$\frac{1}{\alpha_s(\mu)} - \frac{1}{\alpha_s(\mu')} = b\log\frac{\mu}{\mu'}, \tag{29}$$

and hence

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \alpha_s(\mu_0)b\log\frac{\mu}{\mu_0}}. \tag{30}$$

The Landau pole occurs at

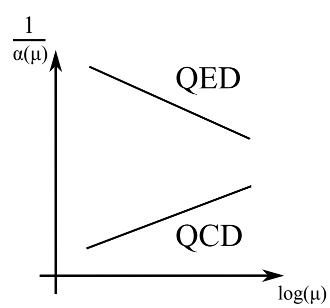


Figure 3: α^{-1} scales linearly with $\log(\mu)$ with a positive gradient in quantum chromodynamics, as with the $g\phi^3$ theory.

$$\alpha_s(\mu_0)b\log\frac{\Lambda_{QCD}}{\mu_0} = -1, \tag{31}$$

and so

$$\Lambda_{QCD} = \mu_0 e^{-\frac{1}{b\alpha_s(\mu_0)}} \approx 250\text{MeV}, \tag{32}$$

independently of our choice of μ_0. Finally, we put our coupling constant in the form

$$\alpha_s(\mu) = \frac{1}{b\log\frac{\mu}{\Lambda_{QCD}}}. \tag{33}$$

Near Λ_{QCD}, quantum chromodynamics becomes strongly coupled. The form of this coupling leads to many interesting phenomena, including confinement, and chiral symmetry breaking: $\left\langle \bar{U}_L U_R \right\rangle \neq 0$.