8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

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Lecture 25

5.3.2: Computation of Beta-Functions

We consider the beta functions for the mass m and the coupling g:

$$\beta_g = \mu \frac{dg}{d\mu}, \quad \beta_m = \mu \frac{d\lambda_m}{d\mu}, \tag{1}$$

where $\lambda_m = \frac{m^2(\mu)}{\mu^2}$. Note that the coupling constants in different renormalization schemes are generally different. In general, we have

$$\begin{aligned} \{\lambda_i\}: &\quad \text{scheme 1}, \\ \{\tilde{\lambda}_i\}: &\quad \text{scheme 2}. \end{aligned}$$

In the problem set, we will see how the $\beta-$functions transform. In particular, the first two terms are universal.

Example 1: $g\phi^3$ in $d = 6$ with MS scheme

$$\begin{aligned} \mathscr{L} &= -\frac{1}{2}(\partial\phi_B)^2 - \frac{1}{2}m_B^2\phi_B^2 + \frac{g_B}{6}\phi_B^3 \\ &= -\frac{1}{2}(1+A)(\partial\phi)^2 - \frac{1}{2}m^2(1+B)\phi^2 + \frac{g}{6}\mu^{\frac{\epsilon}{2}}(1+C)\phi^3, \end{aligned}$$

with

$$\begin{aligned} \phi_B &= (1+A)^{\frac{1}{2}}\phi, \\ m_B &= (1+A)^{-\frac{1}{2}}(1+B)^{\frac{1}{2}}m, \\ g_B &= g\mu^{\frac{\epsilon}{2}}(1+A)^{\frac{3}{2}}(1+C), \end{aligned}$$

$A = -\frac{\alpha}{6\epsilon}$, $B = -\frac{\alpha}{\epsilon}$ and $C = -\frac{\alpha}{\epsilon}$, up to $O(\alpha^2)$. The key is to note that the bare quantities should be independent of μ :

$$\mu\frac{dm_B}{d\mu} = 0, \quad \mu\frac{dg_B}{d\mu} = 0. \tag{2}$$

This leads to results for β_g and β_m. In general, in the case of dimensional regularization and minimal subtraction,

$$g_i^{(B)} = \mu^{\delta_i(\epsilon)}\left[\lambda_i(\mu) + \sum_{n=1}^{\infty}\epsilon^{-n}G_i^{(n)}(\lambda_j)\right] \tag{3}$$

where $\delta_i(\epsilon) = \delta_i + a_i\epsilon$: the last correction is due to dimensional regularization. From (2), we have

$$\beta_i(\epsilon) = \mu\frac{d\lambda_i}{d\mu}. \tag{4}$$

We can expand

$$\beta_i(\epsilon) = \beta_i + \epsilon\alpha_i \tag{5}$$

where the first term is the $\beta-$function and the second term again comes from dimensional regularization. If we take the $\mu-$derivative of (3), we find

$$\begin{aligned} 0 &= \delta_i(\epsilon)\left[\lambda_i + \sum_{i=1}^{\infty}\epsilon^{-n}G_i^{(n)}\right] \\ &\quad + \left[\beta_i(\epsilon) + \sum_{i=1}^{\infty}\epsilon^{-n}\frac{\partial G_i^{(n)}}{\partial\lambda_j}\beta_j(\epsilon)\right]. \end{aligned}$$

Equating both sides of the above equation order by order in ϵ, we find

$$\begin{aligned} (\delta_i + a_i\epsilon)\left[\lambda_i + \epsilon^{-1}G_i^{(0)} + \epsilon^{-2}G_i^{(2)} + \ldots\right] & \\ + \left[\beta_i + \epsilon\alpha_i + \epsilon^{-1}\frac{\partial G_i^{(0)}}{\partial\lambda_j}(\beta_j + \epsilon\alpha_j) + \epsilon^{-2}\ldots\right] &= 0, \end{aligned}$$

and so, at $O(\epsilon)$,

$$\alpha_i = -\lambda_i a_i, \tag{6}$$

(note that we are not invoking the summation convention here,) and, at $O(\epsilon^2)$,

$$\delta_i\lambda_i + a_i G_i^{(1)} + \beta_i + \sum_j \alpha_j \frac{\partial G_i^{(1)}}{\partial\lambda_j} = 0, \tag{7}$$

or, equivalently,

$$\beta_i = -\delta_i\lambda_i - a_i G_i^{(1)} + \sum_j \frac{\partial G_i^{(1)}}{\partial\lambda_j} a_j\lambda_j. \tag{8}$$

We note that the β_i are determined by simple-pole residues of the counter-terms, and that at $O(\epsilon^{-n})$ for $n \geq 1$, the constraints determine $G_i^{(n)}$ for $n \geq 2$ in terms of $G_i^{(1)}$. We now return to our discussion of the $g\phi^3-$theory. Here,

$$\begin{aligned} g_1 &= \alpha_B = \alpha\mu^\epsilon (1+A)^{-3}(1+B), \\ g_2 &= m_B^2 = \mu^2\lambda_m\left(1 - \frac{5\alpha}{6\epsilon} + \ldots\right), \end{aligned}$$

where $\alpha \equiv \frac{g^2}{(4\pi)^3}$, and so we find

$$\delta_1(\epsilon) = \epsilon, \quad \delta_1 = 0, \quad a_1 = 1, \quad G_1^{(1)} = -\frac{3}{2}\alpha^2, \tag{9}$$

and

$$\delta_2(\epsilon) = 2, \quad \delta_2 = 2, \quad a_2 = 0, \quad G_1^{(1)} = -\frac{5}{6}\lambda_m\alpha. \tag{10}$$

From this, we find for the $\beta-$functions,

$$\begin{aligned} \beta_\alpha &= \frac{3}{2}\alpha^2 - 3\alpha^2 = -\frac{3}{2}\alpha^2, \\ \beta_m &= -2\lambda_m - \frac{5}{6}\lambda_m\alpha = -2\lambda_m - \frac{5}{6}\lambda_m\alpha + \ldots. \end{aligned}$$

Let us consider the physical implications of these equations.

1. At weak coupling, $\alpha \ll 1$, β_m is dominated by the first term, $\beta_m \approx -2\lambda_m$. This gives the dimension in the absence of the interaction, which implies the familiar behaviour

$$\lambda_m(\mu) = \lambda_m(\mu_0)\left(\frac{\mu_0}{\mu}\right)^2, \tag{11}$$

and so $\lambda_m(\mu)$ grows quadratically as we decrease μ.

2. α is marginal in the absence of interactions, and so, interactions are important to determine the leading contribution. For $g\phi^3$, $\beta_2 < 0$, and the coupling is marginally relevant: α becomes stronger going into the infrared, as we decrease μ, and stronger going into the ultraviolet, as we increase μ.

We now integrate

$$\mu\frac{d\alpha}{d\mu} = -\frac{3}{2}\alpha^2, \tag{12}$$

which is equivalent to

$$d\left(\frac{1}{\alpha}\right) = \frac{3}{2}d\log\mu. \tag{13}$$

Suppose that $\alpha(\mu_0) = \alpha_0$. Then we have

$$\frac{1}{\alpha(\mu)} - \frac{1}{\alpha_0} = \frac{3}{2}\log\frac{\mu}{\mu_o}, \tag{14}$$

and hence,

$$\alpha(\mu) = \frac{\alpha_0}{1 + \frac{3\alpha_0}{2}\log\frac{\mu}{\mu_0}}. \tag{15}$$

In particular, as $\mu \longrightarrow \infty$, $\alpha(\mu) \longrightarrow 0$. This is asymptotic freedom. $\alpha(\mu) \longrightarrow \infty$ when $\frac{3\alpha_0}{2}\log\frac{\mu}{\mu_0} = -1$. That

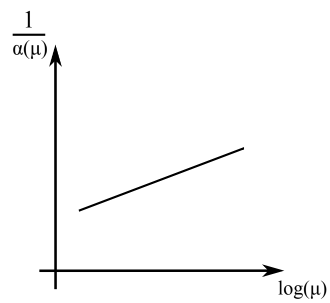

Figure 1: We find a linear relationship between $\alpha^{-1}(\mu)$ and $\log(\mu)$ with a positive gradient for the ϕ^3 theory.

is, when $\mu = \mu_0 e^{-\frac{2}{3\alpha_0}} \equiv \Lambda$. We note that this discussion only applies to $\alpha(\mu) \ll 1$. Of course, our one-loop approximation already breaks down before Λ is reached. Nevertheless, Λ provides a characteristic scale for the system. Λ is independent of μ_0. We can rewrite

$$\alpha(\mu) = \frac{2}{3}\frac{1}{\log\frac{\mu}{\Lambda}}, \tag{16}$$

and instead of specifying $\alpha(\mu_0) = \alpha_0$, we can simply specify Λ. The system does not have any dimensionless coupling. Rather, it has only a scale Λ. This is known as **dimensional transmutation**.

Now let us go back to the issue of the large logarithms encountered in the on-shell scheme when $k^2 \gg m^2$. We encountered $\alpha\log\frac{k^2}{m^2}$ at the one-loop level. However, it goes away if we choose $\mu \sim k$. We want to understand why this is, and why we can still trust perturbation theory. To see what is happening, let us consider, for $\mu \sim k$

$$\alpha(\mu_0 = m) = \alpha_0, \tag{17}$$

so that

$$\alpha(\mu) = \frac{\alpha}{1 + \frac{3}{2}\alpha_0\log\frac{\mu}{m}} \sim \alpha_0\sum_{n=0}^{\infty}\left(\alpha_0\log\frac{\mu}{m}\right)^n. \tag{18}$$

These are exactly the logarithmic terms we have seen before. They were just transferred to the relation between $\alpha(\mu \sim k)$ and α_0. The higher loop correction give higher powers in $\alpha_0\log\frac{\mu}{m}$. As a perturbation series in α_0, the last expression becomes bad when $\alpha_0\log\frac{\mu}{m}$ becomes large, but through the miracle of the renormalization group flow, by integrating the $\beta-$function, we have essentially resummed this bad series as far as $\alpha(\mu)$ remains small for all μ. This remarkable result is the essence of the renormalization group flow, which clearly also applies to the Wilsonian approach.

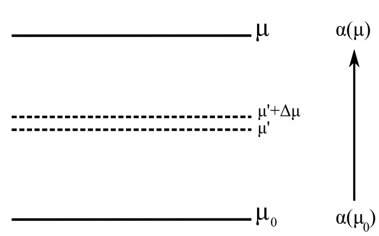

Figure 2: $\alpha(\mu)$ and $\alpha(\mu_0)$ are separated by large logarithms, but if we take infinitesimal steps, $\Delta\alpha = \alpha^2(\mu)\log\frac{\mu+\Delta\mu}{\mu} \sim \alpha^2(\mu)\frac{\Delta\mu}{\mu}$, and so for $\alpha^2(\mu)$ small, we can ignore the higher order terms.

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