8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

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Lecture 24

Let us consider some of the renormalization schemes we discussed in the previous lecture. A particularly convenient renormalization scheme in dimensional regularization is minimal subtraction (MS). In this case, we take $a = b = c = 0$. This gives

$$\begin{aligned} \Pi_{MS}(k^2) &= -\frac{\alpha}{2}\left[\frac{k^2}{6} + m^2 + \int_0^1 dx\, D \log\left(\frac{4\pi\mu^2}{e^\gamma D}\right)\right], \\ \frac{1}{g}V_{MS}(k_1,k_2,k_3) &= 1 + \frac{\alpha}{2}\int dF_3 \log\left(\frac{4\pi\mu^2}{e^\gamma \tilde{D}}\right), \end{aligned}$$

where $\int dF_3 \equiv \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(x_1 + x_2 + x_3 - 1)$, $D \equiv x(1-x)k^2 + m^2$ and $\tilde{D} \equiv m^2 + x_2x_3k_1^2 + x_1x_3k_2^2 + x_1x_2k_3^2$. The k-dependence of $g(\mu)$ and $m(\mu)$ should be such that physical observables are independent of μ. In the $\overline{\text{MS}}$ scheme, we take

$$\frac{1}{\epsilon} \longrightarrow \frac{1}{\epsilon} - \frac{1}{2}\log\left(\frac{4\pi^2}{e^\gamma}\right), \tag{1}$$

giving

$$\Pi_{\overline{MS}}(k^2) = -\frac{\alpha}{2}\left[\frac{k^2}{6} + m^2 + \int_0^1 dx\, D\log\left(\frac{\mu^2}{D}\right)\right]. \tag{2}$$

In the on-shell scheme, we have

$$\mathscr{L}(\phi_B, m_B, g_B) = -\frac{1}{2}Z_\phi^{(on)}\left(\partial\phi_{phys}\right)^2 - \frac{1}{2}Z_m^{(on)}m_{phys}^2\phi_{phys}^2 + \frac{g_{phys}}{6}\mu^{\frac{\epsilon}{2}}Z_g^{(on)}\phi_{phys}^3. \tag{3}$$

In the MS scheme, we have

$$\mathscr{L}(\phi_B, m_B, g_B) = -\frac{1}{2}Z_\phi^{(MS)}\left(\partial\phi_{MS}\right)^2 - \frac{1}{2}Z_m^{(MS)}m_{MS}^2\phi_{MS}^2 + \frac{g_{MS}}{6}\mu^{\frac{\epsilon}{2}}Z_g^{(MS)}\phi_{MS}^3. \tag{4}$$

Finally, in the $\overline{\text{MS}}$ scheme, we have

$$\mathscr{L}(\phi_B, m_B, g_B) = -\frac{1}{2}Z_\phi^{(\overline{MS})}\left(\partial\phi_{\overline{MS}}\right)^2 - \frac{1}{2}Z_m^{(\overline{MS})}m_{\overline{MS}}^2\phi_{\overline{MS}}^2 + \frac{g_{\overline{MS}}}{6}\mu^{\frac{\epsilon}{2}}Z_g^{(\overline{MS})}\phi_{\overline{MS}}^3. \tag{5}$$

For the fields, we have

$$\begin{aligned} \phi_B &= \left(Z_\phi^{(on)}\right)^{\frac{1}{2}}\phi_{phys} \\ &= \left(Z_\phi^{(MS)}\right)^{\frac{1}{2}}\phi_{MS} \\ &= \left(Z_\phi^{(\overline{MS})}\right)^{\frac{1}{2}}\phi_{\overline{MS}}. \end{aligned}$$

The three field renormalizations here are all divergent, but their ratios are finite. Why do we not just use the on-shell scheme?

1. There are instances where it can't be used, for example, if $m_{phys} = 0$.

2. Other schemes can be more convenient in certain settings.

3. More seriously, consider $\left|k^2\right| \gg m^2$. Then we have

$$D \approx x(1-x)k^2, \tag{6}$$

and

$$\log \frac{D}{D_0} \approx \log \frac{k^2}{m^2} + \ldots. \tag{7}$$

Hence,

$$\Pi(k^2) \approx \frac{\alpha}{12} k^2 \log\left(\frac{k^2}{m^2}\right), \tag{8}$$

which can be large compared with k^2, and so perturbation theory is no longer a good approximation. Similarly,

$$\frac{1}{g} V(k_1, k_2, k_3) \approx 1 + \alpha \log\left(\frac{k^2}{m^2}\right), \tag{9}$$

and perturbation theory is not a good approximation for large k^2. Introducing μ allows us to address this problem: if we choose $\mu \sim k$, no such logarithm arises.

If we choose μ appropriately, that is, to be comparable to the momentum scale of the physical process, we can improve our perturbation expansion. As we will see shortly,

1. $g(\mu)$ and $m(\mu)$ can be considered as the counterparts of the scale-dependent coupling constants of the Wilsonian approach.

2. The reason we get large logarithmic terms in the on-shell scheme is that we are trying to use coupling defined at one scale to describe physics at very different scales. We will return to this point later with a physical explanation.

Let us consider the structure of general correlation functions. Having looked at $\Pi(k^2)$ and $V(k_1, k_2, k_3)$ at the one-loop level, let us now look at general connected Greens functions in some renormalization scheme, such as the MS scheme:

$$G_n(x_1, \ldots, x_n) = \langle \Omega \left| T(\phi(x_1) \ldots \phi(x_n)) \right| \Omega \rangle, \tag{10}$$

where $\phi(x)$ is the renormalized field. Then

$$G_n(\{x\}, g(\mu), m(\mu); \mu) = G_n(\{x\}, \lambda_i(\mu); \mu) \tag{11}$$

where $\lambda_i(\mu)$ are dimensionless coupling corresponding to g and m^2, defined with respect to μ. For example, $\lambda_m = \frac{m^2(\mu)}{\mu^2}$. We consider also

$$G_n^{(B)}(x_1, \ldots, x_n) = \langle \Omega \left| T(\phi_B(x_1) \ldots \phi_B(x_n) \right| \Omega \rangle. \tag{12}$$

Then

$$G_n^{(B)}(\{x\}, g_B, m_B; \Lambda_0), \tag{13}$$

where Λ_0 is an ultraviolet cut-off, independent of μ. Since $\phi_B = Z_\phi^{\frac{1}{2}} \phi$, we have that

$$G_n^{(B)} = Z_\phi^{\frac{n}{2}} G_n, \tag{14}$$

and so

$$\mu \frac{d}{d\mu} (Z_\phi^{\frac{n}{2}} G_n) = 0. \tag{15}$$

Introducing $\gamma \equiv \frac{1}{2} \mu \frac{d}{d\mu} \log Z_\phi$, we have

$$\left(\mu \frac{d}{d\mu} + n\gamma\right) G_n = 0, \tag{16}$$

that is,

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_i \frac{\partial}{\partial \lambda_i} + n\gamma\right) G_n = 0 \tag{17}$$

where $\beta_i \equiv \mu \frac{d\lambda_i}{d\mu}$. This is the Callan-Symanzik equation. We note that $\gamma = \gamma(\{\lambda_i\})$ and $\beta_i = \beta_i\left(\{\lambda_i\}\right)$. From the Callan-Symanzik equation, we can express $G_n(\{x\}, \lambda_i(\mu'); \mu')$ in terms of $G_n(\{x\}, \lambda_i(\mu); \mu)$. First, consider $\gamma = 0$. Then we have

$$\frac{d}{d\mu} G_n = 0, \tag{18}$$

and so

$$G_n(\{x\}, \lambda_i(\mu'); \mu') = G_n(\{x\}, \lambda_i(\mu); \mu). \tag{19}$$

These are the running coupling constants we had earlier. In the case $\gamma \neq 0$, γ does not depend on μ explicitly, but it does indirectly through the $\gamma = \gamma\left(\lambda_j\left(\mu\right)\right)$. In this case, we have

$$G_n(\{x\}, \lambda_i(\mu'); \mu') = \exp\left[-n \int_{\log \mu}^{\log \mu'} d\xi \, \gamma\left(\lambda_j\left(\xi\right)\right)\right] \times G_n(\{x\}, \lambda_i(\mu); \mu). \tag{20}$$

γ captures how the definition of ϕ changes as we change μ. In the case that there is only one coupling g, that is, if $m = 0$, then since $\mu \frac{dg}{d\mu} = \beta$,

$$d\xi = \frac{dg}{\beta}, \tag{21}$$

$$G_n(\{x\}, \lambda_i(\mu'); \mu') = \exp\left[-n \int_{\log \mu}^{\log \mu'} \frac{dg'}{\beta(g')} \gamma\left(g'\right)\right] \times G_n(\{x\}, \lambda_i(\mu); \mu). \tag{22}$$

Let us consider some applications of this:

1. For the high momentum behviour, consider

$$G_2\left(p, \lambda(\mu'); \mu'\right) = \eta^{-2}(\mu', \mu) G_2\left(p, \lambda_i(\mu); \mu\right) \tag{23}$$

where $\eta^{-n} \equiv \exp\left[-n \int_{\log \mu}^{\log \mu'} d\xi \, \gamma\left(\lambda_j\left(\xi\right)\right)\right]$. In particular,

$$G_2\left(\kappa p, \lambda(\kappa\mu'); \kappa\mu'\right) = \eta^{-2}(\kappa\mu, \mu) G_2\left(\kappa p, \lambda_i(\mu); \mu\right). \tag{24}$$

This gives a one-dimensional group,

$$G_2\left(p, \lambda_i(\mu); \mu\right) = \frac{1}{p^2} f_2\left(\frac{p}{\mu}, \lambda_i(\mu)\right) \tag{25}$$

and so

$$\begin{aligned} G_2\left(\kappa p, \lambda_i(\kappa\mu); \kappa\mu\right) &= \frac{1}{\kappa^2 p^2} f_2\left(\frac{p}{\mu}, \lambda_i(\kappa\mu)\right) \\ &= \frac{1}{\kappa^2} G_2\left(p, \lambda_i(\kappa\mu); \mu\right). \end{aligned}$$

Equivalently, we have

$$G_2\left(\kappa p, \lambda_i(\mu); \mu\right) = \frac{\eta^2(\kappa\mu, \mu)}{\kappa^2} G_2\left(p, \lambda_i(\kappa\mu); \mu\right). \tag{26}$$

2. At a fixed point, $\beta_i = 0$, and so λ_i is a constant, so $\gamma(\{\lambda_i\})$ is also a constant, and

$$\left(\mu \frac{d}{d\mu} + n\gamma\right) G_n(\{x\}) = 0. \tag{27}$$

Consider, for example, $G_2(x; \mu) = \mu^2 f(\mu x)$. Then we have that

$$\left(y \frac{d}{dy} + 2\Delta\right) f(y) = 0, \tag{28}$$

where $\Delta = 2 + \gamma$. Hence,

$$f(y) = \frac{c}{y^{2\Delta}} \tag{29}$$

and

$$G_2(x;\mu) = \frac{c'}{x^{2\Delta}}. \tag{30}$$

More generally, for

$$G_n\left(\{x\}\,;\mu\right) = \mu^{n\Delta_0} f_n\left(\{\mu x\}\right) \tag{31}$$

where Δ_0 is the canonical dimension, from the Callan-Symanzik equation, we have that f_n should satisfy

$$f_n\left(\{\lambda y\}\right) = \lambda^{-n\Delta} f_n\left(\{y\}\right) \tag{32}$$

with $\Delta = \Delta_0 + \gamma$.

In summary, we have introduced the renormalization scale μ, and $\lambda_i(\mu)$ are scale-dependent couplings, given by the renormalization group flow. Different choices of μ correspond to different descriptions of physical observables. However, the physical observables themselves do not depend on the choice of μ.

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