8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

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Lecture 22

Firstly, we will summarize our previous results. We start with a bare Lagrangian,

$$\mathscr{L}[\Lambda_0, \phi] = \sum_i g_i^{(0)} O_i. \tag{1}$$

The path integral

$$Z = \int_{k<\Lambda_0} \mathfrak{D}\phi \, e^{-\int d^d x \mathscr{L}[\Lambda_0]} \tag{2}$$

describes all the physics below the cutoff Λ_0. Most often we are interested in physics at some energy scale $E \ll \Lambda_0$. $\mathscr{L}[\Lambda_0]$ is not convenient to use, as it contains the degrees of freedom $\phi(k) : E < k < \Lambda_0$ which are not directly related to the physics at the scale E. However, we cannot simply discard them, as they have indirect effects, which can be taken into account by integrating them out. We write $\phi(k) = \phi_\Lambda(k < \Lambda) + \tilde{\phi}(\Lambda < k < \Lambda_0)$, and

$$Z_0 = \int_{k<\Lambda} \mathfrak{D}\phi_\Lambda(k) \int_{\Lambda<k<\Lambda_0} \mathfrak{D}\tilde{\phi}(k) \, e^{-S\left[\phi_\Lambda + \tilde{\phi}, \Lambda_0\right]} = \int_{k<\Lambda} \mathfrak{D}\phi_\Lambda(k) \, e^{-S[\phi_\Lambda, \Lambda]}, \tag{3}$$

where $S[\phi_\Lambda, \Lambda] = \int d^d x \sum_i g_i(\Lambda) O_i,\ g_i = g_i\left(\left\{g_i^{(0)}\right\}, \Lambda_0; \Lambda\right)$. By varying Λ, we obtain a continuous family of $S[\phi_\Lambda, \Lambda]$ or $\{g_i(\Lambda)\}$. This is the renormalization group flow.

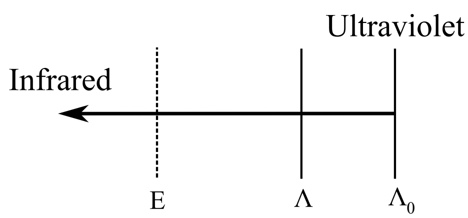

Figure 1: The renormalization flow from the cutoff Λ_0 in the ultraviolet to a cutoff Λ to study processes at an energy scale E in the infrared region.

Infinitesimally, we have

$$\Lambda \frac{dS_\Lambda}{d\Lambda} = F(S_\Lambda), \quad \Lambda \frac{d\lambda_i}{d\Lambda} = \beta_i\left(\{\lambda_j(\Lambda)\}\right), \tag{4}$$

where $\lambda_i = g_i \Lambda^{-\delta_i}$. The process means that all S_Λ should describe the same low-energy physics. By dimensional analysis, we expect that for $\frac{\Lambda}{\Lambda_0} \ll 1$,

$$\lambda_i(\Lambda) \sim \lambda_i(\Lambda_0) \left(\frac{\Lambda}{\Lambda_0}\right)^{-(d-\Delta_i)}, \tag{5}$$

and so we have three cases:

$$\lambda_i : \begin{cases} \Delta_i < d & \text{relevant,} \\ \Delta_i = d & \text{marginal,} \\ \Delta_i > d & \text{irrelevant.} \end{cases} \tag{6}$$

We expect the initial values of irrelevant couplings should not be important for $\frac{\Lambda}{\Lambda_0} \longrightarrow 0$. This rough argument can be substantiated by analyzing the flow equation in detail. It turns out that the flow equation can be written in a closed form, as we showed in the last lecture:

$$S[\phi_\Lambda, \Lambda] = S_0[\phi_\Lambda, \Lambda] + S_{int}[\phi_\Lambda, \Lambda], \tag{7}$$

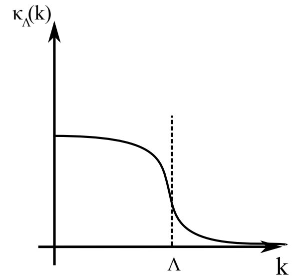

Figure 2: The propagator $G_\Lambda(k) = \frac{1}{k^2}\kappa_\Lambda(k)$ has a cutoff around $k \sim \Lambda$.

where $S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\phi(k)\phi(-k)G_\Lambda^{-1}(k)$. If $G_\Lambda = \frac{1}{k^2}$, we have $S_0 = \frac{1}{2}\int d^4x\,(\partial\phi)^2$. We considered the case $G_\Lambda = \frac{1}{k^2}\kappa_\Lambda(k)$, where κ provides a cut-off at $k \sim \Lambda$. An example is a sharp cutoff $\kappa_\Lambda(k) = \Theta(1-\frac{k}{\Lambda})$. This means $\phi(k)$ with $k > \Lambda$ do not propagate, and there is no need to impose $k < \Lambda$ explicitly in the path integral. Requiring the partition function to be independent of the choice of Λ led to the equation

$$\Lambda\frac{d}{d\Lambda}S_I = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\Lambda\frac{dG_\Lambda(k)}{d\Lambda}\left[\frac{\delta S_I}{\delta\phi(k)}\frac{\delta S_I}{\delta\phi(-k)} - \frac{\delta^2 S_I}{\delta\phi(k)\delta\phi(-k)}\right]. \tag{8}$$

Remarks:

1. This equation is exact, and fully non-perturbative.

2. $\Lambda\frac{dG_\Lambda}{d\Lambda}$ is only supported near a thin shell of momentum around Λ. In fact, for $G_\Lambda = \Theta(1-\frac{k}{\Lambda})$, we have $\Lambda\frac{dG_\Lambda}{d\Lambda} \propto \delta(k-\Lambda)$. Physically, this reflects that the flow equation is obtained by integrating out the degrees of freedom around Λ.

3. Expanding $S_I[\phi,\Lambda]$ in momentum space as

$$S_I[\phi,\Lambda] = \sum_{n=2}^{\infty}\frac{1}{n!}\int\left(\prod_{i=1}^{n}\frac{d^dk_i}{(2\pi)^4}\right)(2\pi)^4\delta^{(d)}(k_1+k_2+\ldots+k_n)\times g(k_1,\ldots,k_n;\Lambda)\phi(k_1)\ldots\phi(k_n). \tag{9}$$

(8) requires that

$$\Lambda\frac{d}{d\Lambda}g(k_1,\ldots k_n;\Lambda) = \sum_{\{I_1,I_2\}} g(-p,I_1;\Lambda)\Lambda\frac{dG_\Lambda(p)}{d\Lambda}g(p,I_2;\Lambda) - \frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Lambda\frac{dG_\Lambda}{d\Lambda}g(q,-q,k_1,\ldots,k_n;\Lambda) \tag{10}$$

where $p = \sum_{k_i\in I_1}k_i$, and I_1, I_2 are disjoint subsets of momenta such that $I_1\cup I_2 = \{k_1,\ldots,k_n\}$. $\sum_{\{I_1,I_2\}}$ is a sum over all possible ways to separate $\{k_1,\ldots,k_n\}$ into groups. Diagramatically, this is shown in figure 3.

This corresponds to integrating out a tree-level diagram and a one-loop momentum diagram respectively.

3. We can expand $S_I[\phi,\Lambda]$ in coordinate space:

$$S_I[\phi,\Lambda] = \int d^4x\,\sum_i g_i(\Lambda)O^i(x), \tag{11}$$

where the O_i form a complete set of local operators. From (8), we obtain

$$\Lambda\frac{dg_i}{d\Lambda} = \tilde{\beta}_i^{\,j}g_j + \tilde{\beta}_i^{\,jk}g_jg_k, \tag{12}$$

with $\tilde{\beta}_i^{\,i} = 0$. Using dimensionless couplings, $\lambda_i(\Lambda) = g_i(\Lambda)\Lambda^{-(d-\Delta_i)}$, we find

$$\beta_i \equiv \Lambda\frac{d\lambda_i}{d\Lambda} = \beta_i^{\,j}\lambda_j + \beta_i^{\,jk}\lambda_j\lambda_k, \tag{13}$$

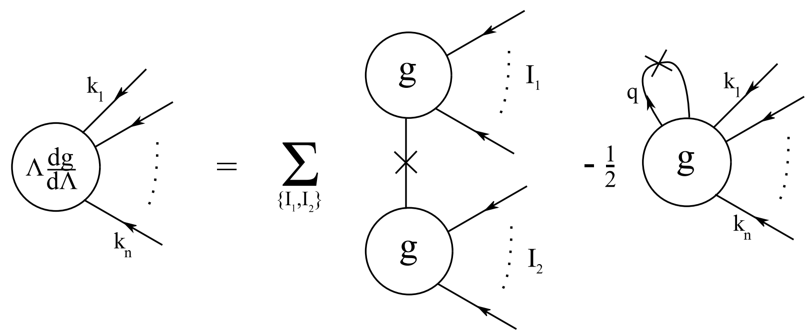

Figure 3: Diagramatic representation of (10), where the crossed vertex represents $\Lambda \frac{dG_\Lambda}{d\Lambda}(p)$.

where $\beta_i^j = (\Delta_i - d)\,\delta_i^j + \tilde{\beta}_i^j$, and the $\tilde{\beta}_i^j$ has no diagonal term. We see how this corresponds to the cases

$$\begin{cases} \Delta_i > d & \text{damping,} \\ \Delta_i = 0 & \text{marginal,} \\ \Delta_i < d & \text{growth.} \end{cases} \tag{14}$$

4. The flow equation, (8), and thus the resulting β-functions, are not unique. It can be written in many other, equivalent forms by using field redefinitions:

$$\phi(x) \longrightarrow \phi(x) + a\phi^2(x) + b\phi^3(x) + c\,(\partial\phi)^2 + \ldots. \tag{15}$$

Such field redefinitions lead to redefinitions of the couplings, but they should not change the physical observables. There is also a scheme dependence on the choice of cutoff functions, $\kappa_\Lambda(k)$. The equations (12) and (13) are an infinite number of coupled first-order differential equations. It is quite complicated to analyze them, and often requires the development of approximation methods.

Let us consider some general features of the flow:

1. Separate the couplings, as before, as $\{\lambda_i\} = \{\rho_a\} + \{\kappa_\alpha\}$, where $\{\rho_a\}$ are the relevant and marginal couplings, and $\{\kappa_\alpha\}$ are the irrelevant couplings. Then, for a generic theory with all $\lambda_i^{(0)} \sim O(1)$ at Λ_0, we have for $\frac{\Lambda}{\Lambda_0} \ll 1$, assuming the $\{\lambda_i\}$ do not become too large, that the $\{\kappa_\alpha\}$ only depend on the $\{\rho_a\}$. For example, if we consider two couplings, λ_4 and λ_6, for terms of the form ϕ^4 and ϕ^6 respectively, we have

$$\begin{aligned} \Lambda \frac{d\lambda_4}{d\Lambda} &= \lambda_6 + \ldots, \\ \Lambda \frac{d\lambda_6}{d\Lambda} &= 2\lambda_6 - \lambda_4^2 + \ldots \end{aligned}$$

The first term in the flow equation for λ_6 provides a damping when going into the infrared regime.

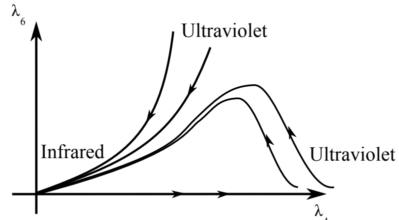

Figure 4: Damping and renormalization flow into the point $\lambda_4 = \lambda_6 = 0$ for the irrelevant coupling λ_6 and the marginally irrelevant coupling λ_4.

as $\frac{\Lambda}{\Lambda_0} \to 0$, $\lambda_6 \to \lambda_6 = \lambda_6(\lambda_4) \approx \frac{\lambda_4^2}{2}$. The flow of λ_4 is given by $\beta_4 = \Lambda \frac{d\lambda_4}{d\Lambda} = \frac{\lambda_4^2}{2} + O(\lambda_4^3)$, and so λ_4 is marginally irrelevant.

We can now make the connection to the standard renormalization procedure. We consider the ϕ^4-Lagrangian,

$$\mathscr{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_{phys}^2\phi^2 - \frac{1}{4}\lambda_4^{(phys)}\phi^4 + \mathscr{L}_{ct}, \tag{16}$$

where m^2 and λ_4 are renormalized physical quantities which can be measure experimentally. They should be interpreted as being defined at a specific infrared scale. For example,

$$\lambda_4^{(phys)} = \lambda_4(\Lambda = 0). \tag{17}$$

We now keep $\lambda_4^{(phys)}$ and m_{phys}^2 fixed and take the limit of the cutoff $\Lambda_0 \longrightarrow \infty$. All physical observables only depend on the renormalized quantities. In particular, $\lambda_4^{(0)}(\Lambda_0) \neq 0$, $\lambda_6^{(0)} = 0$. That is, we start along the horizontal axis. The Wilsonian approach tells us that such an initial condition is not important. Also note that λ_6 is not zero in the infrared, it is just determined by λ_4. λ_6 is related to six-particle scatterings, which of course have non-zero amplitude.

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