8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

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Lecture 21

5.1: RENORMALIZATION GROUP FLOW

Consider the bare action defined at a scale Λ_0 :

$$S[\Lambda_0] = \int^{\Lambda_0} d^d x \, \frac{1}{2} (\partial\phi)^2 + \sum_i g_i^{(0)} O_i, \tag{1}$$

where O_i is a complete set of local operators formed from ϕ. The theory is specified by the set $\left\{g_i^{(0)}\right\}$. As explained in the previous lecture, we can change the cutoff scale to some $\Lambda < \Lambda_0$ by integrating out the degrees of freedom in the interval (Λ, Λ_0). This gives

$$S[\Lambda] = \int^{\Lambda} d^d x \, \frac{1}{2} (\partial\phi)^2 + \sum_i g_i(\Lambda) O_i, \tag{2}$$

after redefining ϕ to absorb the field renormalization factor Z. This theory is specified by the set $\{g_i(\Lambda)\}$. Similarly, at another scale $\Lambda' < \Lambda$, we obtain $S[\Lambda']$, described by $\{g_i(\Lambda')\}$. These three actions, S_{Λ_0}, S_Λ and $S_{\Lambda'}$, should all describe the same physics at an energy scale $E < \Lambda' < \Lambda < \Lambda_0$. The relations between them can be found by integrating out the degrees of freedom explicitly in the path integral, giving

$$\begin{aligned} g_i(\Lambda) &= g_i(g_i^{(0)}, \Lambda_0; \Lambda), \\ g_i(\Lambda') &= g_i(g_i^{(0)}, \Lambda_0; \Lambda') \\ &= g_i'(g_i(\Lambda), \Lambda; \Lambda'). \end{aligned}$$

This process describes the renormalization group transformations, or the renormalization group flow: transformations between couplings at different scales to ensure they describe the same low energy physics. If we consider, for

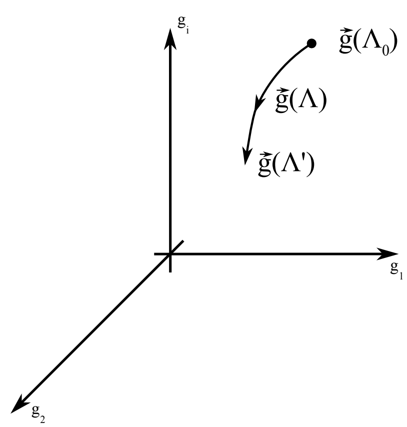

Figure 1: The renormalization flow as the flow in the space of all possible coupling parameterizations to ensure the same low-energy physics at different scales.

simplicity, the dimensionless couplings $\{\lambda_i(\Lambda)\}$ defined by $\lambda_i \equiv g_i \Lambda^{-\delta_i}$, differentiating gives

$$\Lambda \frac{d\lambda_i}{d\Lambda} = \beta_i(\{\lambda_j(\Lambda)\}) \tag{3}$$

where $\beta_i(\{\lambda_j(\Lambda)\}) = \frac{d}{d\ln\Lambda} \lambda_i(\{\lambda_j(\Lambda)\}, Z)\big|_{Z=1}$. It is important to note that the β_i are only functions of the dimensionless coupling constants $\{\lambda_j(\Lambda)\}$: they do not depend on Λ explicitly, as can be seen by considering integrating out a fraction of the highest-energy modes in the path integral. The β-functions give the tangent vector of the flow, and depend only on the values of $\{\lambda_j\}$. Under a relabeling of couplings,

$$\tilde{\lambda}_i = \tilde{\lambda}_i(\{\lambda_j\}), \tag{4}$$

we have that

$$\tilde{\beta}_i(\left\{\tilde{\lambda}\right\}) = \sum_j \frac{d\tilde{\lambda}_i}{d\lambda_j}\beta_j(\{\lambda\}). \tag{5}$$

The $\beta-$functions can be computed explicitly from the path integral:

$$\begin{aligned} Z[J] &= \int_{|k|<\Lambda} \mathfrak{D}\phi\, e^{-S[\Lambda,\phi_\Lambda]-\int d^dx\, J\phi} \\ &= \int_{|k|<\Lambda'} \mathfrak{D}\phi_{\Lambda'}(k) \int_{\Lambda'<|k|<\Lambda} \mathfrak{D}\tilde{\phi}(k)\, e^{-S\left[\phi_{\Lambda'}+\tilde{\phi},\Lambda\right]-\int d^dx\, J(\phi_\Lambda+\tilde{\phi})} \\ &= \int_{|k|<\Lambda} \mathfrak{D}\phi_{\Lambda'}(k)\, e^{-S\left[\phi_{\Lambda'},\Lambda'\right]-\int d^dx\, J\phi_{\Lambda'}}. \end{aligned}$$

Now, if we let $\Lambda' \longrightarrow \Lambda - \delta\Lambda$, $S[\Lambda - \delta\Lambda] = S[\Lambda] + \delta S[\Lambda]$, we have

$$\Lambda \frac{dS_\Lambda}{d\Lambda} = F(S_\Lambda) \tag{6}$$

Expanding

$$S_\Lambda = \sum_i g_i O_i = \sum_i \lambda_i \Lambda^{\delta_i} O_i, \tag{7}$$

(6) gives us the β-functions for all couplings. As an example, let us consider the case of a free scalar field in four dimensions, with a cut-off at a scale Λ. Then, we have

$$S_\Lambda[\phi] = \int_{k<\Lambda} \frac{d^4k}{(2\pi)^4} f(k)\phi^*_\Lambda(k)\phi_\Lambda(k). \tag{8}$$

We expand $f(k)$ as a power series in k:

$$\begin{aligned} f(k) &= m_0^2 + k^2 + r_4 k^4 + \ldots \\ &= \lambda_m(\Lambda)\Lambda^2 + k^2 + \frac{\tilde{r}_4(\Lambda)}{\Lambda^2}k^4 + \ldots, \end{aligned}$$

where the coefficient of k^2 can be chosen to one with a suitable normalization for ϕ_Λ. Here, $\lambda_m(\Lambda)$, $\tilde{r}_4(\Lambda), \ldots$ are dimensionless couplings: $\left[\phi^2\right] = 2$, $\left[\left(\partial^2\phi\right)^2\right] = 6$, and so $\delta_m = 2$, $\delta_{r_4} = -2$, for example. We now let $\phi_\Lambda(k) = \phi_{\Lambda'}(k) + \tilde{\phi}(k)$ with $\tilde{\phi}(k)$ supported for $k \in (\Lambda', \Lambda)$ and $\phi_{\Lambda'}$ supported for $k \in (0, \Lambda')$. Then we have that

$$S_\Lambda[\phi_\Lambda] = S_\Lambda[\phi_{\Lambda'}] + S_\Lambda\left[\tilde{\phi}\right] + 2\int \frac{d^4k}{(2\pi)^4} f(k)\phi_{\Lambda'}(k)\tilde{\phi}(k), \tag{9}$$

where the last term is zero as $\phi_{\Lambda'}$ and ϕ_k have disjoint support. Integrating out $\tilde{\phi}$ only generates an overall constant for the path integral, and so

$$S_{\Lambda'}[\phi_{\Lambda'}] = S_\Lambda[\phi_{\Lambda'}] = \int_{k<\Lambda'} \frac{d^4k}{(2\pi)} f(k)\phi^*_{\Lambda'}(k)\phi_{\Lambda'}(k) \tag{10}$$

where $f(k)$ has not changed. That is,

$$\begin{aligned} f(k) &= m_0^2 + k^2 + r_4 k^4 + \ldots \\ &= \lambda_m(\Lambda')\Lambda'^2 + k^2 + \frac{\tilde{r}_4(\Lambda')}{\Lambda'^2}k^4 + \ldots, \end{aligned}$$

and we conclude that

$$\begin{aligned}\lambda_m(\Lambda') &= \lambda_m(\Lambda)\left(\frac{\Lambda}{\Lambda'}\right)^2 = \lambda_m(\Lambda)\left(\frac{\Lambda}{\Lambda'}\right)^{-\delta_m} \quad \text{is a relevant operator,}\\ \tilde{r}_4(\Lambda') &= \tilde{r}_4(\Lambda)\left(\frac{\Lambda}{\Lambda'}\right)^{-2} = \tilde{r}_4(\Lambda)\left(\frac{\Lambda}{\Lambda'}\right)^{-\delta_m} \quad \text{is an irrelevant operator.}\end{aligned}$$

Similarly,

$$\begin{aligned}\beta_m &= \Lambda' \left.\frac{d\lambda_m(\Lambda')}{d\Lambda'}\right|_{\Lambda'\to\Lambda} = -2\lambda_m = -\delta_m\lambda_m < 0,\\ \beta_{r_4} &= \Lambda' \left.\frac{d\tilde{r}_4(\Lambda')}{d\Lambda'}\right|_{\Lambda'\to\Lambda} = 2\tilde{r}_4 = -\delta_m\tilde{r}_4 > 0.\end{aligned}$$

We note that dimensional quantities like m^2 and r_4 do not change at all in this instance, but that the dimensionless couplings flow as they are defined with respect to the cut-off scale. This does reflect the right physics: the relative importance of each term in $f(k)$ as we go to lower energies, or smaller k. That is,

$$\frac{m_0^2}{k^2} \text{ becomes larger as } k \text{ becomes smaller,}$$
$$\frac{r_4 k^4}{k^2} \text{ becomes smaller as } k \text{ becomes smaller.}$$

We will now derive the full flow equation for $S_\Lambda[\phi]$. For this purpose, we write it as

$$S[\phi,\Lambda] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} G_\Lambda^{-1}(k)\phi(k)\phi(-k) + S_I[\phi,\Lambda] + U(\Lambda) \tag{11}$$

where $U(\Lambda)$ is a cosmological constant, and the propagator $G_\Lambda(k)$ satisfies

$$G_\Lambda(k) = \begin{cases}\frac{1}{k^2} & k \ll \Lambda,\\ 0 & k \gg \Lambda.\end{cases} \tag{12}$$

We have that

$$Z = \int \mathfrak{D}\phi(k)\, e^{-S_0[\phi,\Lambda]-\tilde{S}_I[\phi,\Lambda]}, \tag{13}$$

where $\tilde{S}_I = S_I + U$. There is now no need to impose an explicit cut-off when integrating over $\phi(k)$. It is clearly

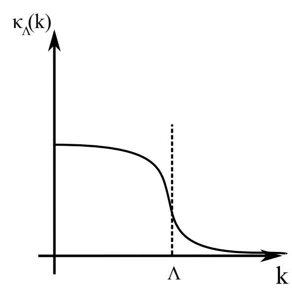

Figure 2: The propagator $G_\Lambda(k) = \frac{1}{k^2}\kappa_\Lambda(k)$ has a cut-off around $k \sim \Lambda$.

very complicated to obtain the flow equation for $\tilde{S}_I[\phi,\Lambda]$ by evaluating the path integral directly. We will instead require

$$\Lambda\frac{dZ[\Lambda]}{d\Lambda} = 0, \tag{14}$$

which is an equivalent statement. From this, we have

$$\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left\langle \phi(-k)\phi(k)e^{-\tilde{S}_I}\right\rangle \Lambda\frac{dG_\Lambda^{-1}}{d\Lambda} = \left\langle \Lambda\partial_\Lambda e^{-\tilde{S}_I}\right\rangle. \tag{15}$$

Here, $\langle \ldots \rangle = \frac{1}{Z_0} \int \mathfrak{D}\phi \ldots e^{-S_0}$, with $Z_0 = \int \mathfrak{D}\phi \, e^{-S_0}$. We would like to express the left-hand side of (15) more directly in terms of S_I. For this purpose, consider

$$0 = \int \mathfrak{D}\phi \, \frac{\delta}{\delta\phi(k)} \left(\phi(k) e^{-S_0 - \tilde{S}_I} \right). \tag{16}$$

From this, we have that

$$(2\pi)^4 \, \delta^{(4)}(0) \left\langle e^{-\tilde{S}_I} \right\rangle - G_\Lambda^{-1} \left\langle \phi(k)\phi(-k) e^{-\tilde{S}_I} \right\rangle + \left\langle \phi(k) \frac{\delta}{\delta\phi(k)} e^{-\tilde{S}_I} \right\rangle = 0. \tag{17}$$

The last term in this equation is still complicated. Consider further

$$0 = \int \mathfrak{D}\phi \, \frac{\delta^2}{\delta\phi(k)\delta\phi(-k)} \left(e^{-S_0 - \tilde{S}_I} \right). \tag{18}$$

From this, we have

$$(2\pi)^4 \, \delta^{(4)}(0) G_\Lambda^{-1} \left\langle e^{-\tilde{S}_I} \right\rangle - \left(G_\Lambda^{-1} \right)^2 \left\langle \phi(k)\phi(-k) e^{-\tilde{S}_I} \right\rangle - 2G_\Lambda^{-1} \left\langle \phi(k) \frac{\delta}{\delta\phi(k)} e^{-\tilde{S}_I} \right\rangle + \left\langle \frac{\delta^2}{\delta\phi(k)\delta\phi(-k)} e^{-\tilde{S}_I} \right\rangle = 0. \tag{19}$$

If we multiply 17 by $2G_\Lambda^{-1}$ and add the result to (19), we obtain

$$(2\pi)^4 \, \delta^{(4)}(0) G_\Lambda^{-1} \left\langle e^{-\tilde{S}_I} \right\rangle - \left(G_\Lambda^{-1} \right)^2 \left\langle \phi(k)\phi(-k) e^{-\tilde{S}_I} \right\rangle + \left\langle \frac{\delta^2}{\delta\phi(k)\delta\phi(-k)} e^{-\tilde{S}_I} \right\rangle = 0. \tag{20}$$

Eliminating $\left\langle \phi(k)\phi(-k) e^{-\tilde{S}_I} \right\rangle$ between (15) and (20) gives

$$\begin{aligned} \left\langle \Lambda \frac{d}{d\Lambda} e^{-S_I - U} \right\rangle &= -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \, \Lambda \frac{dG_\Lambda}{d\Lambda} \left\langle \frac{\delta^2}{\delta\phi(k)\delta\phi(-k)} e^{-S_I - U} \right\rangle \\ &\quad -\frac{1}{2} (2\pi)^4 \, \delta^{(4)}(0) \int \frac{d^4k}{(2\pi)^4} \, \Lambda \frac{d \log G_\Lambda}{d\Lambda} \left\langle e^{-S_I - U} \right\rangle. \end{aligned}$$

Here, the second term is a constant, and so we have

$$\Lambda \frac{d}{d\Lambda} U = \frac{1}{2} V_4 \Lambda \frac{d}{d\Lambda} \int \frac{d^4k}{(2\pi)^4} \, \log G_\Lambda(k), \tag{21}$$

where $V_4 = (2\pi)^4 \, \delta^{(4)}(0)$, and so

$$U(\Lambda) = U_0 + \frac{1}{2} V_4 \int \frac{d^4k}{(2\pi)^4} \, \log G_\Lambda(k) \tag{22}$$

where U_0 is independent of Λ, and

$$\Lambda \frac{d}{d\Lambda} e^{-S_I} = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \, \Lambda \frac{dG_\Lambda(k)}{d\Lambda} \frac{\delta^2}{\delta\phi(k)\delta\phi(-k)} e^{-S_I}, \tag{23}$$

or, equivalently,

$$\Lambda \frac{d}{d\Lambda} S_I = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \, \Lambda \frac{dG_\Lambda(k)}{d\Lambda} \left[\frac{\delta S_I}{\delta\phi(k)} \frac{\delta S_I}{\delta\phi(-k)} - \frac{\delta^2 S_I}{\delta\phi(k)\delta\phi(-k)} \right]. \tag{24}$$

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