

8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

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Lecture 19

The procedure described for quantum electrodynamics last lecture can be readily generalized to other theories:

$$D = d - \sum_f E_f \Delta_f - \sum_i V_i \delta_i, \quad (1)$$

with $\Delta_f > 0$. For a renormalizable theory, all $\delta_i \geq 0$. That is, all the interactions are renormalizable. This means only for a finite set of classes of diagrams characterized by $\{E_f\}$ is $D \geq 0$, and D is always bounded by d . Therefore, all superficial divergences can be canceled by a finite set of counterterms of the form

$$\sum_{j=0}^D A_j \partial^j \prod_f \phi_f^{E_f} \quad (2)$$

where the A_j are divergent constants. From the facts that $\Delta_f > 0$ and $D \geq 0$, this set of interactions are all renormalizable: $[A_i] \geq 0$. Therefore, all divergences can be absorbed into redefinitions of the couplings, masses, and fields, provided the Lagrangian contains all the renormalizable interactions allowed by symmetries, in order to make sure the overlapping divergences cancel.

Remarks:

1. For quantum electrodynamics, there is no need to enumerate all the diagrams. We only need to write down all possible counterterms consistent with gauge symmetry.
2. Non-renormalizable theories, with some $\delta_i < 0$, require an infinite number of counterterms.
3. In a super-renormalizable theory, each counterterm contains only a finite number of divergent terms, when expanded in perturbation theory.
4. Generally any divergence allowed by symmetries will occur.
5. If the regulator for ultraviolet divergences does not violate a given symmetry, the divergences violating that symmetry will not occur. This is particularly important and subtle for gauge symmetries, because of the need of gauge fixing. For quantum electrodynamics, the Ward identities ensure this. For non-Abelian gauge theories, the generalized Ward identities arising from the BRST symmetry ensure this.
6. There are situations in which one cannot find any regulators which preserve a symmetry, and so the symmetry can be violated when the regulator is removed. This leads to the concept of an anomaly: a symmetry of the classical Lagrangian which is not respected in quantum theory. If this happens to a gauge symmetry, it leads to an inconsistency.

4.2.3: Ultraviolet Divergences and Renormalization in Coordinate Space

Our discussion has been so far restricted to momentum space, with loop momenta as $p \rightarrow \infty$, leading to ultraviolet divergences. One would normally expect from that uncertainty principle that ultraviolet divergences are associated with short-distance phenomena. This can be made manifest by calculating the Feynman diagrams in coordinate space, leading to the result that counterterms must be local. This also leads to the result that in systems at a finite temperature T , the counterterms are independent of T . We will not give a complete treatment, but we will briefly highlight the local nature of the counterterms. We will use a scalar field for illustration. In the free theory,

$$\mathcal{L}_0 = -\frac{1}{2} (\partial\phi)^2 - \frac{1}{2} m^2 \phi^2. \quad (3)$$

If we try to compute operators like $\phi^2(x)$, $\phi^3(x)$, etc., we find that they are not well-defined, containing ultraviolet divergences. For example, consider

$$\phi(x)\phi(y) = G(x-y) + : \phi(x)\phi(y) : \quad (4)$$

where the first term is the contraction and the second term is the normal ordering. As $x \rightarrow y$, the first term is singular: $G(x-y) \sim \frac{1}{(x-y)^{d-2}}$. The second term is regular in this limit. Hence,

$$\phi^2(x) = C_0 1 + : \phi^2(x) : , \quad (5)$$

where $C_0 \sim \frac{1}{\epsilon^{d-2}}$, for a short-distance regulator ϵ . Similarly,

$$\phi^4(x) = \tilde{C}_0 1 + C_1 : \phi^2(x) : + : \phi^4(x) : . \quad (6)$$

In the interacting theory, we consider, for example,

$$\mathcal{L}_{int} = \lambda \phi^4, \quad (7)$$

or more generally

$$\mathcal{L}_{int} = \lambda O, \quad (8)$$

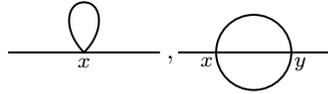
where O is some composite operator built from ϕ . The general physical amplitude is given, in the interaction picture, by

$$\langle T(X) \rangle = \left\langle T(X e^{i \int d^d x \lambda O(x)}) \right\rangle_0 \quad (9)$$

where X is a collection of fields (operators), and the right-hand side is evaluated in the free theory. At the n th order in perturbation theory, the contribution is

$$\lambda^n \int \prod_{i=1}^n d^d x_i \langle T(X O(x_1) \dots O(x_n)) \rangle_0 . \quad (10)$$

There are two types of divergent loops. Considering, for example, $O = \phi^4$:



$$\quad (11)$$

The first type corresponds to a single propagator attached to the same vertex, and the second case corresponds to $O(x)O(y)|_{x \rightarrow y}$. The first type of divergence can be taken care of by normal ordering O itself:

$$\phi^4(x) = \tilde{C}_0 1 + C_1 : \phi^2(x) : + : \phi^4(x) : . \quad (12)$$

Here, the first term is canceled by Λ_0 , and the second term is cancelled by a mass renormalization. The second type of divergence can also be dealt with using normal ordering:

$$: \phi^4(x) : : \phi^4(y) : |_{x \rightarrow y} = \frac{1}{\epsilon^8} 1 + \frac{1}{\epsilon^2} : \phi^2(y) : + \frac{1}{\epsilon^4} : (\partial\phi)^2(y) : + \frac{1}{\epsilon^4} : \phi^4(y) : + \frac{1}{\epsilon^2} : \phi^6(y) : + \dots \quad (13)$$

Here, $\epsilon = |x-y|$ is a short-distance regulator. The first four terms are not integrable, but can again be canceled by adding

$$A_0 + A_1 \phi^2 + A_3 \phi^4 + A_2 (\partial\phi)^2 \quad (14)$$

counterterms to the Lagrangian. In general,

$$: O(x) : : O(y) : |_{x \rightarrow y} = \sum_n C_n(\epsilon) : O_n(y) : , \quad (15)$$

where the $C_n(\epsilon) \sim \frac{1}{\epsilon^\alpha}$ are ordinary functions, and the $: O_n(y) :$ are normal-ordered local operators. Here,

$$\alpha = 2\Delta - \Delta_n, \quad (16)$$

where $[O] = \Delta$ and $[O_n] = \Delta_n$. This expansion is known as the operator product expansion, or the OPE. The structure of the OPE is such that the counterterms are always local. The OPE will be discussed in more detail in the problem set.

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