8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

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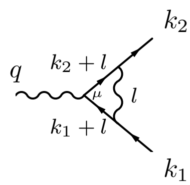
Lecture 16

Firstly, we summarize the results of the vertex correction from the previous lecture:

$$\Gamma_1^\mu(k_1,k_2) \equiv$$

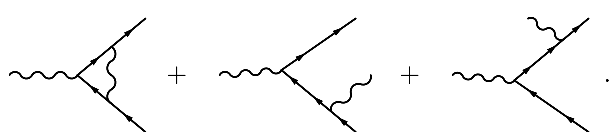


$$\begin{aligned} &= \gamma^\mu A(q^2) + i(k_1+k_2)^\mu B(q^2) \\ &= e\gamma^\mu F_1(q^2) - \frac{\sigma^{\mu\nu}q_\nu F_2(q^2)}{2m}, \end{aligned} \tag{1}$$

where $eF_1 = A + 2mB$ and $eF_2 = -2mB$. We showed that $F_2(0) = -\frac{2m}{e}B(0) = \frac{\alpha}{2\pi} = 0.0011614..$, where $\alpha \equiv \frac{e^2}{4\pi} \approx \frac{1}{137}$. The integral for Γ_1^μ is, in fact, infrared divergent. As $l \longrightarrow 0$,

$$\begin{aligned} (k_1+l)^2 + m^2 &= k_1^2 + m^2 + 2k_1.l + l^2 \\ &= 2k_1.l + l^2 \longrightarrow 0, \end{aligned}$$

and so $\Gamma_1 \sim \int d^4l \frac{1}{l^4}$ is divergent. This is due to soft photon interactions, and this effect is in fact cancelled by including the soft emissions:



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The explanation is that it is only reasonable to calculate measurable cross-sections:

$$\left(\frac{d\sigma}{d\Omega}\right)_{measured} = \left(\frac{d\sigma}{d\Omega}\right)(\alpha \to \beta) + \left(\frac{d\sigma}{d\Omega}\right)(\alpha \to \beta + \text{soft photons}). \tag{3}$$

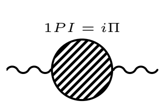
The calculation then proceeds by imposing an infrared cut-off λ on the photon momenta. The divergences in the $\lambda \longrightarrow 0$ limit cancel among virtual and real soft photon emissions, and we can safely take the $\lambda \longrightarrow 0$ limit in the end.

3.4: VACUUM POLARIZATION

We will now evaluate the one-loop correction to the photon propagator, and consider the physical interpretation, recalling the general structure we considered in lecture 12.

3.4.1: One-loop correction

$$i\Pi^{\mu\nu}(k) =$$



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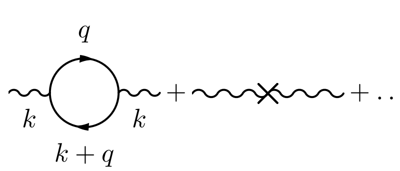


$$= (-1)(-ie)^2 \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\left(\gamma^\mu S_0(k+q)\gamma^\nu S_0(q)\right) - i(Z_3-1)k^2 P_T^{\mu\nu} \tag{4}$$

We note that the factor of (-1) at the front comes from the fermionic loop, that the trace is over the omitted spinor indices, and that S_0 is the electron propagator. Having now enough experience with one-loop diagrams, we will omit the details of the calculation and only emphasise the new aspects. Using $S_0(q) = \frac{1}{i\not{q}-m} = -\frac{i\not{q}+m}{q^2+m^2}$, we first introduce the Feynman parameters:

$$\Pi_1^{\mu\nu} = e^2 \int_0^1 dx \int \frac{d^4p}{(2\pi)^4} \frac{4N^{\mu\nu}}{(p^2+D)^2} + \text{counterterm}, \tag{5}$$

with $p \equiv q + xk$, $D \equiv x(1-x)k^2 + m^2 - i\epsilon$, and

$$\begin{aligned} 4N^{\mu\nu} &= \mathrm{tr}\left[\gamma^\mu(i\not{k}+i\not{q}+m)\gamma^\nu(i\not{q}+m)\right] \\ &= \mathrm{tr}\left[\gamma^\mu(i\not{p}+i(1-x)\not{k}+m)\gamma^\nu(i\not{p}-ix\not{k}+m)\right] \\ &= -\mathrm{tr}\left[\gamma^\mu\not{p}\gamma^\nu\not{p}\right] + m^2\mathrm{tr}\left[\gamma^\mu\gamma^\nu\right] + x(1-x)\mathrm{tr}\left[\gamma^\mu\not{k}\gamma^\nu\not{k}\right] \\ &\quad +\text{terms linear in } p + \text{terms with an odd number of } \gamma \text{matrices.} \end{aligned}$$

We note that the trace of a term with an odd number of γ-matrices gives zero, and that

$$\begin{aligned} \mathrm{tr}\left[\gamma^\mu\gamma^\nu\right] &= 4\eta^{\mu\nu}, \\ \mathrm{tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}). \end{aligned}$$

Hence, disregarding irrelevant terms, we can write

$$N^{\mu\nu} = -2p^\mu p^\nu + 2x(1-x)k^\mu k^\nu + (m^2 + p^2 - x(1-x)k^2)\eta^{\mu\nu}. \tag{6}$$

Secondly, we evaluate the integrals, extending to a general dimension d. We note that

$$\int \frac{d^dp}{(2\pi)^d} p^\mu p^\nu f(p^2) = \frac{\eta^{\mu\nu}}{d} \int \frac{d^dp}{(2\pi)^d} p^2 f(p^2), \tag{7}$$

and so

$$i\Pi^{\mu\nu}(k) = 4e^2 \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{A\eta^{\mu\nu} + p^2(1-\frac{2}{d})\eta^{\mu\nu} + 2x(1-x)k^\mu k^\nu}{(p^2+D)^2}, \tag{8}$$

where we have set $A \equiv m^2 - x(1-x)k^2$. We note that the first and third terms in the numerator are logarithmically divergent, and the second term is quadratically divergent. We now apply a Wick rotation $p^0 \to ip_E^d$, $d^dp \to id^dp_E$ and $p^2 \to p_E^2$. We recall that

$$\int \frac{d^dp_E}{(2\pi)^d} \frac{(p_E^2)^a}{(p_E^2+D)^b} = \frac{\Gamma(b-a-\frac{d}{2})\Gamma(a+\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(b)\Gamma(\frac{d}{2})} D^{-(b-a-\frac{d}{2})}, \tag{9}$$

and so

$$(1-\frac{2}{d}) \int \frac{d^dp_E}{(2\pi)^d} \frac{p_E^2}{(p_E^2+D)^2} = -D \int \frac{d^dp_E}{(2\pi)^d} \frac{1}{(p_E^2+D)^2}. \tag{10}$$

Hence, the numerator of (8) can be replaced by

$$(A-D)\eta^{\mu\nu} + 2x(1-x)k^\mu k^\nu. \tag{11}$$

The transverse component, therefore, is given by

$$-2x(1-x)k^2 P_T^{\mu\nu}, \tag{12}$$

and we can write

$$\begin{aligned} i\Pi^{\mu\nu}(k) &= -8ie^2k^2P_T^{\mu\nu}(k) \int_0^1 dx \int \frac{d^dp_E}{(2\pi)^d} \frac{1}{(p_E^2+D)^2} - i(Z_3-1)k^2P_T^{\mu\nu} \\ &= -8ie^2k^2P_T^{\mu\nu}(k) \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \frac{x(1-x)}{D^{2-\frac{d}{2}}} - i(Z_3-1)k^2P_T^{\mu\nu}. \end{aligned}$$

Thirdly, we use dimensional regularization, setting $d = 4 - \epsilon$, $e \to e\mu^{\frac{\epsilon}{2}}$,

$$i\Pi^{\mu\nu}(k) = \frac{-8ie^2k^2P_T^{\mu\nu}(k)}{16\pi^2}\int_0^1 dx\, x(1-x)\left[\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi\mu^2}{D}\right)\right] - i(Z_3-1)k^2P_T^{\mu\nu}, \tag{13}$$

and so

$$\Pi^{\mu\nu}(k) = k^2P_T^{\mu\nu}\Pi(k^2), \tag{14}$$

with

$$\Pi(k^2) = -\frac{e^2}{\pi^2}\int_0^1 dx\, x(1-x)\frac{1}{\epsilon} - \frac{1}{2}\log\left(\frac{D}{4\pi\mu^2e^{-\gamma}}\right) - (Z_3-1). \tag{15}$$

The physical field condition constrains that $\Pi(k^2 = 0) = 0$, and so Z_3 is fixed by

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\frac{1}{\epsilon} - \frac{1}{2}\log\left(\frac{m^2}{4\pi\mu^2e^{-\gamma}}\right) \tag{16}$$

and the final result for $\Pi(k^2)$ is

$$\Pi(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\log\left(1 + \frac{k^2x(1-x)}{m^2}\right) \tag{17}$$

Remarks:

1. For $k^2 < -4m^2$, $\Pi(k^2)$ becomes complex.

2. For more than one charged particle, we must add their respective contributions, and the smallest m contributes most.

3. The internal propagator is given by

$$\begin{aligned} e^2D^{\mu\nu}(q) &\to \frac{-ie^2}{q^2-i\epsilon}\frac{\eta^{\mu\nu}}{1-\Pi(q^2)} \\ &= \frac{-ie^2\eta^{\mu\nu}}{q^2-i\epsilon}(1+\Pi(q^2)+\ldots). \end{aligned}$$

We see that for $q^2 \gg m^2$, a large spacelike momentum, from (17) we have

$$\begin{aligned} \Pi(q^2) &\approx \frac{e^2}{2\pi^2}\log\frac{q^2}{m^2}\int_0^1 dx\, x(1-x) \\ &= \frac{\alpha}{3\pi}\log\frac{q^2}{m^2}, \end{aligned}$$

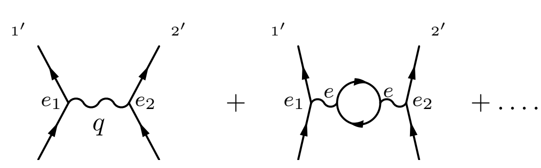
where $\alpha = \frac{e^2}{4\pi}$. Then, the internal propagator goes as

$$\frac{e^2}{q^2-i\epsilon} \longrightarrow \frac{e^2}{1-\Pi(q^2)}\frac{1}{q^2-i\epsilon} \equiv \frac{e^2(q)}{q^2-i\epsilon}, \tag{18}$$

with $e^2(q) = \frac{e^2}{1-\frac{\alpha}{3\pi}\log\frac{q^2}{m^2}}$ the running coupling constant.

3.4.2: Physical implication

Consider scattering of two charged particles with coupling constants e_1 and e_2, for example, in the process $e^- + \mu^- \longrightarrow e^- + \mu^-$:



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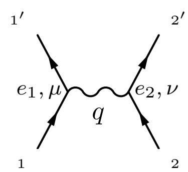
to lowest order, where e is the coupling constant associated with the lightest intermediate particle.

$$S(1,2 \longrightarrow 1',2') \quad = \quad (-ie)^2 \bar{u}_{1'}(p_1')\gamma^\mu u_1(p_1) D_{\mu\nu}(q) \bar{u}_{2'}(p_2')\gamma^\nu u_2(p_2),$$

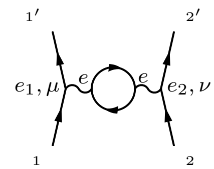
where

$$D_{\mu\nu}(q) = \frac{-i}{q^2 - i\epsilon} \frac{P^T_{\mu\nu}(q)}{1-\Pi(q^2)} + D^L_{\mu\nu}(q), \tag{20}$$

and $P^T_{\mu\nu} = \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$, $q = p_1' - p_1 = p_2' - p_2$. Note that $\bar{u}_{2'}(p_2')\gamma^\nu u_2(p_2) q_\nu = \bar{u}_{2'}(p_2')(\not{p}_2' - \not{p}_2) u_2(p_2) = 0$. We now want to consider corrections to the Coulomb potential. We consider the low energy, non-relativistic limit, where we derive most of our intuition about electromagnetism from, and where the notion of a potential makes most sense. The lowest two diagrams drawn above correspond to



$$= \frac{e_1 e_2}{q^2}\eta_{\mu\nu},$$



$$= \frac{e_1 e_2}{q^2}\Pi(q^2)\eta_{\mu\nu}. \tag{21}$$

In the non-relativistic limit,

$$\left|q^0\right| \sim |v\vec{q}| \ll |\vec{q}|, \quad q^2 \approx \vec{q}^2. \tag{22}$$

In this case, one-photon exchange corresponds to the Born approximation:

$$\frac{e_1 e_2}{\vec{q}^2} = \int d^3\vec{r}\, e^{-i\vec{q}.\vec{r}} V_0(\vec{r}), \tag{23}$$

where $V_0(\vec{r}) = \frac{e_1 e_2}{4\pi|\vec{r}|}$ is the Coulomb potential. The one-loop correction is given by $\frac{e_1 e_2}{\vec{q}^2}\Pi(\vec{q}^2)$, where, from (17)

$$\Pi(\vec{q}^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\log\left(1 + \frac{\vec{q}^2}{m^2}x(1-x)\right), \tag{24}$$

and the term provides a correction to the Coulomb potential

$$\delta V(\vec{r}) = e_1 e_2 \int d^3\vec{r}\, e^{-i\vec{q}.\vec{r}} \frac{\Pi(\vec{q}^2)}{\vec{q}^2}. \tag{25}$$

From now on, we will for convenience write $q \equiv |\vec{q}|$, $r \equiv \|\vec{r}\|$. The angular integral is given by

$$\begin{aligned} &\frac{2\pi}{(2\pi)^3}\int_0^\pi d\theta \sin\theta e^{iqr\cos\theta} \\ &= \frac{1}{4\pi^2}\frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right), \end{aligned}$$

and so we find for the correction to the Coulomb potential

$$\begin{aligned} \delta V(\vec{r}) &= \frac{e_1 e_2}{4\pi^2}\int_0^\infty dq\, \frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right)\Pi(\vec{q}^2) \\ &= \frac{e_1 e_2}{4\pi^2}\int_{-\infty}^\infty dq\, \frac{1}{iqr} e^{iqr}\Pi(\vec{q}^2). \end{aligned}$$

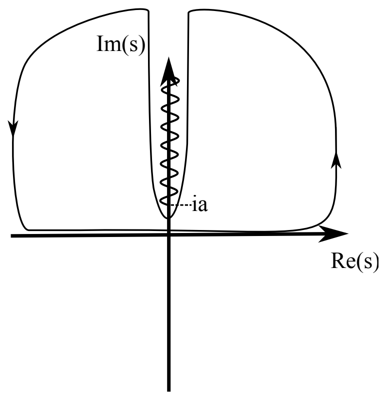


Figure 1: Clockwise contour for the integral in dz in (26). The semi-circular arc is taken to infinity and gives a vanishing contribution.

If we now let $z \equiv qr$, $a \equiv \frac{mr}{\sqrt{x(1-x)}} \geq 2mr$, the result reduces to

$$\delta V(\vec{r}) = \frac{e_1 e_2}{8\pi^4} \frac{e^2}{ir} \int_0^1 dx\, x(1-x) \int_{-\infty}^{\infty} dz\, \frac{e^{iz}}{z} \log\left(1 + \frac{z^2}{a^2}\right). \tag{26}$$

The integral in dz, $I = \int_{-\infty}^{\infty} dz\, \frac{e^{iz}}{z} \log\left(1 + \frac{z^2}{a^2}\right)$, can be computed using the complex contour shown in figure 1, giving

$$\begin{aligned} I &= 2\int_a^\infty d\lambda \frac{e^{-\lambda}}{\lambda} i\pi \\ &= 2i\pi \int_1^\infty d\lambda\, \frac{e^{-a\lambda}}{\lambda} \end{aligned}$$

after setting $\lambda \to a\lambda$. So, our result for the correction to the Coulomb potential is given by

$$\begin{aligned} \delta V(\vec{r}) &= \frac{e_1 e_2}{4\pi} \frac{e^2}{\pi^2} \int_0^1 dx\, x(1-x) \int_1^\infty \frac{d\lambda}{\lambda}\, e^{-\lambda \frac{mr}{\sqrt{x(1-x)}}} \\ &\equiv \frac{e_1 e_2}{4\pi} Z(mr) \end{aligned}$$

Remarks:

1. When $mr \gg 1$, we can evaluate the integral in the saddle-point approximation, using integration by parts,

$$\delta V(r) = \frac{e_1 e_2}{4\pi} \frac{e^2}{16\pi^{\frac{3}{2}}} \frac{e^{-2mr}}{(mr)^{\frac{3}{2}}} + \ldots. \tag{27}$$

2. $Z(mr)$ increases with decreasing r. As $r \longrightarrow 0$, $I(r) \longrightarrow \infty$. If we consider putting a short-distance cut off at $mr = e \ll 1$, we find

$$Z(\epsilon) = \frac{e^2}{6\pi^2} \log\frac{1}{\epsilon} + \ldots, \tag{28}$$

and thus

$$\begin{aligned} V(r) &= V_0 + \delta V(r) \\ &= \frac{e_1 e_2}{4\pi r}(1 + Z(mr)) \\ &= \frac{\tilde{e}_1(r)\tilde{e}_2(r)}{4\pi r}, \end{aligned}$$

where

$$\tilde{e}_i(r) = e_i\left(1 + Z(mr)\right)^{\frac{1}{2}}. \tag{29}$$

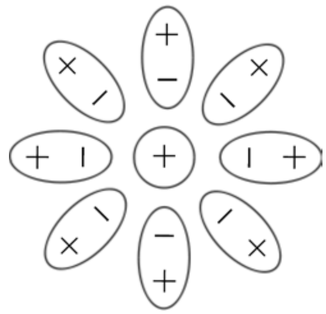


Figure 2: Virtual electron-positron pairs form a screening effect.

We observe that $\tilde{e}_i(r) \longrightarrow \infty$ as $r \longrightarrow 0$, and $\tilde{e}_i(r) \longrightarrow e_i$ as $r \longrightarrow 0$, with small experimental corrections. Physically, we can view $\tilde{e}_i(r = \epsilon)$ as the bare charge, which is very large. The physical interpretation is that virtual electron-positron pairs screen the charge more at large distances. The screening length is given by $r_s \sim \frac{1}{2m}$. That is, there is a negative cloud of size $\frac{1}{2m}$. At large distances, the charges scale as $e_i \sim e^{-2mr}$.

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