8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

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Lecture 13

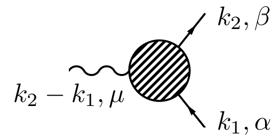
We continue our analysis of renormalization in quantum electrodynamics from last lecture.

3.1.4: Charge Renormalization

Consider the vertex corrections:

$$G_{\mu\alpha\beta}(k_1,k_2) \equiv \int d^4y_1 d^4y_2\, e^{ik_1\cdot y_1 - ik_2\cdot y_2} \langle 0| T(A_\mu(0)\psi_\alpha(y_1)\bar{\psi}_\beta(y_2))|0\rangle$$

$$= \qquad \qquad (1)$$

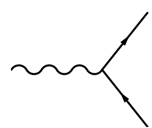


where, again, these are defined in terms of bare quantities. We introduce an effective vertex, Γ, defined by

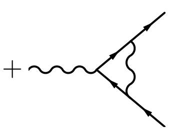
$$G_{\mu\alpha\beta}(k_1,k_2) = D_{\mu\nu}(k_2-k_1)S_{\alpha\delta}(k_1)\Gamma^\nu_{\delta\lambda}(k_1,k_2)S_{\lambda\beta}(k_2). \qquad (2)$$

In perturbation theory,

$$\Gamma^\mu_{\alpha\beta}(k_1,k_2) =$$



$$+$$



$$+\ldots$$

$$= -ie_B\gamma^\mu_{\alpha\beta} + \ldots. \qquad (3)$$

We note that only 1PI diagrams contribute, by the definition. We will now show that gauge invariance, in the form of the Ward identities, puts important constraints on the structure of $\Gamma^\mu_{\alpha\beta}$. Acting on the generating functional for connected diagrams, and setting $J_\mu = \eta = \bar{\eta} = 0$, we have

$$\frac{1}{\xi}\partial^2\partial^\mu \left.\frac{\delta^3 W}{\delta J_\mu(x)\delta\bar{\eta}_\alpha(y_1)\delta\eta_\beta(y_2)}\right|_{J=\eta=\bar{\eta}=0} = ie_B\left[\delta^{(4)}(x-y_1)\frac{\delta^2 W}{\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y_2)} - \delta^{(4)}(x-y_2)\frac{\delta^2 W}{\delta\bar{\eta}_\alpha(y_1)\delta\eta_\beta(x)}\right]_{J=\eta=\bar{\eta}=0}, \qquad (4)$$

or, equivalently,

$$\frac{1}{\xi}\partial^2\partial^\mu \langle 0|T(A_\mu(0)\psi_\alpha(y_1)\bar{\psi}_\beta(y_2))|0\rangle = e_B\left[\delta^{(4)}(x-y_1)\langle 0|T(\psi_\alpha(x)\bar{\psi}_\beta(y_2))|0\rangle - \delta^{(4)}(x-y_2)\langle 0|T(\psi_\alpha(y_1)\bar{\psi}_\beta(x))|0\rangle\right]. \qquad (5)$$

Changing basis to momentum space, we have $\partial^\mu \longrightarrow iq^\mu$, where $q^\mu \equiv (k_2-k_1)^\mu$. We can set $x=0$ by applying $\int d^4y_1 d^4y_2\, e^{ik_1\cdot y_1 - ik_2\cdot y_2}$ on both sides, giving

$$-\frac{i}{\xi}q^2 q^\mu D_{\mu\nu}(q)S_{\alpha\delta}(k_1)\Gamma^\nu_{\delta\lambda}(k_1,k_2)S_{\lambda\beta}(k_2) = e_B\left[S_{\alpha\beta}(k_2) - S_{\alpha\beta}(k_1)\right]. \qquad (6)$$

In the last lecture, we showed $\frac{1}{\xi}\partial^2\partial^\mu D_{\mu\nu} = -k_\nu$. And so, the result, when written in terms of matrices in spinor space, reduces to

$$-S(k_1)(q_\nu\Gamma^\nu)S(k_2) = e_B\left(S(k_2) - S(k_1)\right), \qquad (7)$$

or, equivalently,

$$q_\nu\Gamma^\nu(k_1,k_2) = e_B\left(S^{-1}(k_2) - S^{-1}(k_1)\right), \qquad (8)$$

where $q \equiv k_2 - k_1$. This is an important constraint. To see the implications, we consider $k_1 = k$, k on-shell and $q \longrightarrow 0$, meaning k_2 is also close to on-shell. Then

$$S^{-1}(k_1) \approx -\frac{1}{Z_2}(i\not{k}_1 + m - i\epsilon) + \ldots$$

$$S^{-1}(k_2) \approx -\frac{1}{Z_2}(i\not{k}_2 + m - i\epsilon) + \ldots$$

where m here is the physical mass. We then have

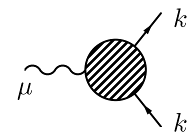
$$q_\nu \Gamma^\nu(k,k) = -\frac{e_B}{Z_2} i\not{q}, \tag{9}$$

or

$$\Gamma^\nu(k,k) = -\frac{ie_B}{Z_2}\gamma^\nu \tag{10}$$

when k is on shell. The physical charge we measure should be

$$\Gamma^\mu_{phys}(k,k) \quad = $$



$$\equiv \quad -ie_{phys}\gamma^\mu. \tag{11}$$

where k is on-shell and we are using the physical fields. In other words, consider

$$G^{(phys)}_\mu(k_1,k_2) = \int d^4y_1 d^4y_2\, e^{ik_1\cdot y_1 - ik_2\cdot y_2} \langle 0|\, T(A_\mu(0)\psi_\alpha(y_1)\bar{\psi}_\beta(y_2))\, |0\rangle \tag{12}$$

where the fields are now the physical fields, rather than the bare fields. Then

$$G^{(phys)}_\mu(k_1,k_2) = D_{\mu\nu}(q)S(k_1)\Gamma^\nu_{phys}(k_1,k_2)S(k_2) \tag{13}$$

where $D_{\mu\nu}$ and S are again here for the physical fields. Since

$$A^B_\mu = \sqrt{Z_3}A_\mu, \quad D^B_{\mu\nu} = Z_3 D_{\mu\nu}, \quad \psi^B = \sqrt{Z_2}\psi, \quad S^B = Z_2 S, \tag{14}$$

we have that

$$G^B_\mu = \sqrt{Z_3}(\sqrt{Z_2})^2 G^{(phys)}_\mu, \tag{15}$$

where $G^B \equiv D^B S^B \Gamma^B S^B$ and $G^{(phys)} = DS\Gamma^{(phys)}S$. From this, we have that

$$\Gamma^\nu(k,k) = \sqrt{Z_3}Z_2\Gamma^\nu_B(k,k) \tag{16}$$

and so

$$e = \sqrt{Z_3}e_B. \tag{17}$$

The dependence of e on Z_2 cancels precisely as a result of $\Gamma_B \propto \frac{e_B}{Z_2}$. That $\frac{e}{e_B} = \sqrt{Z_3}$ only depends on Z_3, the field strength renormalization of the photon, has important implications: the ratio is universal for all charged fields. Suppose that $e_B^{proton} = e_B^{electron}$. Then it is necessarily true that $e^{proton} = e^{electron}$, despite the proton and electron interacting very differently and having different masses. If $\frac{e}{e_B}$ depended on Z_2, for example, then we would have an extremely difficult time in explaining why $e^{proton} = e^{electron}$, as their respective values of Z_2 are very different. Finally, in terms of renormalized quantities:

$$q_\nu\Gamma^\nu(k_1,k_2) = e\left[S^{-1}(k_2) - S^{-1}(k_1)\right]. \tag{18}$$

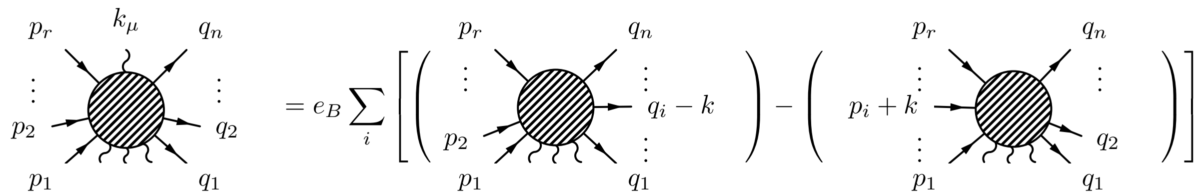
We note additionally that for k_1 and k_2 on-shell, but $k_1 \neq k_2$,

$$q_\nu\Gamma^\nu(k_1,k_2) = 0. \tag{19}$$

This is an example of a large class of identities. These are known as the general Ward identities. These identities are obtained by acting on the generating functional for connected diagrams with

$$\frac{\delta}{\delta J_{\nu_1}(z_1)}\frac{\delta}{\delta J_{\nu_2}(z_2)}\cdots\frac{\delta}{\delta\bar{\eta}(y_1)}\frac{\delta}{\delta\bar{\eta}(y_2)}\cdots\frac{\delta}{\delta\eta(x_1)}\frac{\delta}{\delta\eta(x_2)}\cdots, \tag{20}$$

and then setting $J_\mu = \bar{\eta} = \eta = 0$. The resulting expression is most transparently written diagramatically in momentum space:



(21)

where each external line should be considered as an exact photon or fermion propagator, except that associated with k_μ on the left-hand side, which should be amputated.

Remarks:

1. Suppose we attach to Γ^μ a photon propagator at the q^ν. If all the external propagators are on-shell, the longitudinal part of the propagator does not contribute as the inner product of this with q^ν is zero. This enforces gauge invariance.

2. Suppose we attach to Γ^μ an external photon line, that is, $\epsilon_\mu \Gamma^\mu(k,\ldots)$. This is invariant under $\epsilon_\mu \longrightarrow \epsilon_\mu + k_\mu$, which is a gauge transformation $A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$.

3. Only charged particles need to be on-shell. Other photon lines or any other neutral particles (if they exist) can be off-shell, since they they do not transform under gauge transformations.

4. For Ward identities to be valid, regularization should preserve gauge invariance.

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