8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

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Lecture 12

3: GENERAL ASPECTS OF QUANTUM ELECTRODYNAMICS

3.1: RENORMALIZED LAGRANGIAN

Consider the Lagrangian of quantum electrodynamics in terms of the bare quantities:

$$\mathscr{L} = -\frac{1}{4}F^B_{\mu\nu}F^{\mu\nu}_B - i\bar{\psi}_B(\gamma^\mu(\partial_\mu - ie_B A^B_\mu) - m_B)\psi_B. \tag{1}$$

We use the convention:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{2}$$

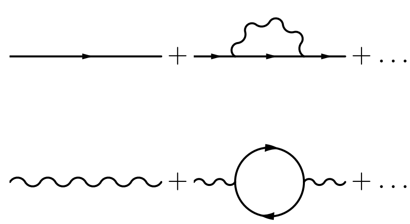
$$\gamma_0^2 = -1, \quad \gamma_0^\dagger = -\gamma_0, \quad \gamma_i^\dagger = \gamma_i,$$
$$\bar{\psi} = \psi^\dagger\gamma_0, \quad \not{k} = k^\mu\gamma_\mu, \quad \not{k}^2 = k^2.$$

where, in four dimensions, $(\eta^{\mu\nu}) = \text{diag}(-1, 1, 1, 1)$. We expect to find the mass and field renormalizations.

Note: we will omit the "B" signifying bare quantities in what follows.

ψ :

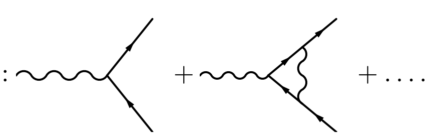
A_μ :



(3)

along with vertex corrections

e :



(4)

We will look at how to introduce renormalized quantities.

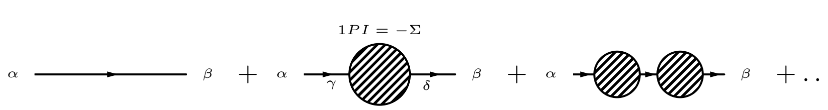
3.1.1: Fermion self-energy

We have that

$$S_{\alpha\beta}(x) = \langle 0| T(\psi_\alpha(x)\bar{\psi}_\beta(0)\,|0\rangle\,, \tag{5}$$

and

$S_{\alpha\beta}(k) =$

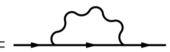


$$\begin{aligned} &= S_0(k) + S_0(-\Sigma)S_0 + S_0(-\Sigma)S_0(-\Sigma)S_0 + \ldots \\ &= S_0\frac{1}{1+\Sigma S_0}, \end{aligned} \tag{6}$$

where we have omitted the spinor indices in the second and third lines: these are to be read as matrix equations. Hence, we have that

$$S^{-1} = S_0^{-1} + \Sigma. \tag{7}$$

Recall that

$$S_0 = \frac{-1}{i\not{k} + m_B}, \quad -\Sigma =$$  $$+ \ldots, \tag{8}$$

and so

$$S^{-1} = -(i\not{k} + m_B) + \Sigma, \tag{9}$$

and we have for the fully interacting two-point function

$$S = \frac{-1}{i\not{k} + m_B - i\epsilon - \Sigma}. \tag{10}$$

Note that $\Sigma = \Sigma(\not{k})$, since it can only depend on $\not{k}$ and k^2. Even though it is a function of matrix, we can treat it as an ordinary function. The physical mass is defined by $\not{k} = im$, so that

$$-m + m_B - \Sigma(im) = 0. \tag{11}$$

We note that, again, Σ will be divergent. Near the pole, we have that

$$S \approx \frac{-Z_2}{i\not{k} + m - i\epsilon} \tag{12}$$

with $Z_2^{-1} = 1 + i\left.\frac{d\Sigma}{d\not{k}}\right|_{\not{k}=im}$. The relations between the bare and physical quantities are given by

$$m_B = m + \delta m, \quad \psi_B = \sqrt{Z_2}\psi \tag{13}$$

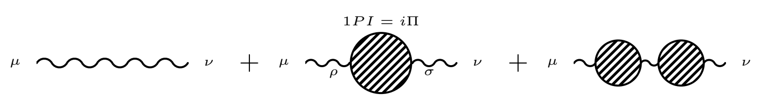
where $\delta m = \Sigma(im)$ and ψ is the renormalized field.

3.1.2: Photon self-energy

Similarly, we have that

$$D_{\mu\nu}(x) = \langle 0 | T(A_\mu^B(x) A_\nu^B(0) | 0 \rangle, \tag{14}$$

and

$$D_{\mu\nu}(k) =$$ 

$$+ \ldots$$

$$\begin{aligned} &= D_0(k) + D_0(i\Pi)D_0 + D_0(i\Pi)D_0(i\Pi)D_0 + \ldots \\ &= D_0 \frac{1}{1 - i\Pi D_0} \end{aligned} \tag{15}$$

Hence,

$$(iD)^{-1} = (iD_0)^{-1} - \Pi. \tag{16}$$

Recall that

$$\begin{aligned} iD_0^{\mu\nu} &= \frac{1}{k^2 - i\epsilon}\left[\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right] \\ &= \frac{1}{k^2 - i\epsilon}\left[P_T^{\mu\nu} + \xi P_L^{\mu\nu}\right], \end{aligned}$$

where we have defined the transverse projector $P_T^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$, and the longitudinal projector $P_L^{\mu\nu} \equiv \frac{k^\mu k^\nu}{k^2}$. Note that it is not a coincidence that the propagator can be built from these two tensors, $\eta^{\mu\nu}$ and $k^\mu k^\nu$: they are the only two two-tensors allowed by symmetry. These projectors satisfy the properties

$$P_T^{\mu\nu} P_{\nu\lambda}^T = P_{T\,\nu}^{\mu}, \quad P_L^{\mu\nu} P_{\nu\lambda}^L = P_{L\,\lambda}^{\mu}, \quad P_T^{\mu\nu} P_{\nu\lambda}^L = 0. \tag{17}$$

Note: T and L are just labels here, and the placing of these indices does not carry meaning. Hence, we have that

$$(iD_0)^{-1} = k^2 \left[P_T^{\mu\nu} + \frac{1}{\xi} P_L^{\mu\nu} \right]. \tag{18}$$

We may also expand $\Pi^{\mu\nu}$ as

$$\begin{aligned} \Pi^{\mu\nu} &= P_T^{\mu\nu} f_T(k^2) + P_L^{\mu\nu} f_L(k^2) \\ &= \eta^{\mu\nu} f_T + \frac{k^\mu k^\nu}{k^2}(f_L - f_T). \end{aligned} \tag{19}$$

Therefore,

$$(iD)^{-1} = P_T^{\mu\nu}(k^2 - f_T) + P_L^{\mu\nu}(\frac{k^2}{\xi} - f_L), \tag{20}$$

and we have for the full interacting photon two-point function,

$$D = -i \left[P_T^{\mu\nu} \frac{1}{k^2 - f_T} + P_L^{\mu\nu} \frac{1}{\frac{k^2}{\xi} - f_L} \right]. \tag{21}$$

We observe that if $f_{T,L}(k^2 = 0) \neq 0$, a mass will be generated for the photon. Because $\Pi^{\mu\nu}$ comes from 1PI diagrams, it should not be singular at $k^2 = 0$, and so $f_L - f_T = O(k^2)$, as $k \longrightarrow 0$. We will show that gauge invariance ensures that no mass is generated from the loop corrections.

3.1.3: Ward identities

Consider the path integral for the generating functional:

$$Z[J_\mu, \eta, \bar{\eta}] = \int \mathfrak{D}A_\mu \mathfrak{D}\psi \mathfrak{D}\bar{\psi} e^{iS[A_\mu, \psi, \bar{\psi}]} \tag{22}$$

where $S = S_{QED} + \int d^4x\, J_\mu A_B^\mu + \bar{\eta}\psi_B + \bar{\psi}_B \eta$, where we note explicitly these couplings are in terms of bare quantities.

$$\mathscr{L}_{QED} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - i\bar{\psi}(\gamma^\mu D_\mu - m)\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{23}$$

We define the generating functional for connected diagrams, $W[J_\mu, \eta, \bar{\eta}]$, by

$$Z[J_\mu, \eta, \bar{\eta}] = e^{iW[J_\mu, \eta, \bar{\eta}]}. \tag{24}$$

For example,

$$\begin{aligned} \langle 0| T(\psi_\alpha(x)\bar{\psi}_\beta(y)) |0\rangle &= i \left. \frac{\delta^2 W[J_\mu, \eta, \bar{\eta}]}{\delta\eta_\alpha(x)\delta\eta_\beta(y)} \right|_{J=\eta=\bar{\eta}=0}, \\ \langle 0| T(A_\mu^B(x) A_\nu^B(y)) |0\rangle &= i \left. \frac{\delta^2 W[J_\mu, \eta, \bar{\eta}]}{\delta J^\mu(x)\delta J^\nu(y)} \right|_{J=\eta=\bar{\eta}=0}. \end{aligned}$$

Recall, for infinitesimal gauge transformations, $\delta A_\mu = \partial_\mu \lambda$, $\delta\psi = ie_B\lambda\psi$, and $\delta\bar{\psi} = -ie_B\lambda\bar{\psi}$. Consider a change of variables in the path integral:

$$\begin{aligned} A_\mu &\longrightarrow A'_\mu = A_\mu + \delta A_\mu, \\ \psi &\longrightarrow \psi' = \psi + \delta\psi, \\ \bar{\psi} &\longrightarrow \bar{\psi}' = \bar{\psi} + \delta\bar{\psi}. \end{aligned}$$

Then we have

$$\int \mathfrak{D}A'_\mu \mathfrak{D}\psi' \mathfrak{D}\bar{\psi}' e^{iS[A'_\mu, \psi', \bar{\psi}']} = \int \mathfrak{D}A_\mu \mathfrak{D}\psi \mathfrak{D}\bar{\psi} e^{iS[A_\mu, \psi, \bar{\psi}]}, \tag{25}$$

as this is just of a change of the dummy integration variables. Note that the measure is unchanged by this shift:

$$\mathfrak{D}A'_\mu \mathfrak{D}\psi' \mathfrak{D}\bar{\psi}' = \mathfrak{D}A_\mu \mathfrak{D}\psi \mathfrak{D}\bar{\psi}, \tag{26}$$

and the action for the two sets of variables are related by

$$S\left[A'_\mu, \psi', \bar{\psi}'\right] = S\left[A'_\mu, \psi', \bar{\psi}'\right] - \frac{1}{\xi}\int d^4x\, \partial_\mu A^\mu \partial^2 \lambda + \int d^4x\, J_\mu \partial^\mu \lambda + ie_B \lambda \bar{\eta}\psi - ie_B \lambda \bar{\psi}\eta. \tag{27}$$

Hence, we must have

$$\int d^4x\, \lambda(x) \int \mathfrak{D}A_\mu \mathfrak{D}\psi \mathfrak{D}\bar{\psi} e^{iS[A,\psi,\bar{\psi}]} \left[-\frac{1}{\xi}\partial^2 \partial_\mu A^\mu - \partial_\mu J^\mu + ie_B(\bar{\eta}\psi - \bar{\psi}\eta)\right] = 0. \tag{28}$$

Since

$$\begin{aligned} A_\mu(x) &\sim -i\frac{\delta Z}{\delta J^\mu(x)} = Z\frac{\delta W}{\delta J^\mu(x)}, \\ \psi(x) &\sim -i\frac{\delta Z}{\delta \bar{\eta}(x)} = Z\frac{\delta W}{\delta \bar{\eta}(x)}, \\ \bar{\psi}(x) &\sim -i\frac{\delta Z}{\delta \eta(x)} = Z\frac{\delta W}{\delta \eta(x)}, \end{aligned}$$

we have that

$$\frac{1}{\xi}\partial^2 \partial^\mu \left. \frac{\delta^2 W}{\delta J^\mu(x) \delta J^\nu(y)} \right|_{J=\eta=\bar{\eta}=0} + \partial_\nu \delta^{(4)}(x-y) = 0, \tag{29}$$

that is,

$$\frac{i}{\xi}\partial^2 \partial^\mu D_{\mu\nu}(x-y) + \partial_\nu \delta^{(4)}(x-y) = 0, \tag{30}$$

or, written in momentum-space,

$$-\frac{i}{\xi}k^2 k^\mu D_{\mu\nu}(k) + k_\nu = 0. \tag{31}$$

If we now write

$$D_{\mu\nu}(k) = P^T_{\mu\nu} D_T(k^2) + P^L_{\mu\nu} D_L(k^2), \tag{32}$$

with $k^\mu P^L_{\mu\nu} = k_\nu$, the Ward identity reduces to

$$\frac{-i}{\xi}k^2 k_\nu D_L(k^2) + k_\nu = 0, \tag{33}$$

and so

$$D_L(k^2) = -\frac{i\xi}{k^2}, \tag{34}$$

and the longitudinal part of the two-point function is completely determined. Comparing this with (21), we find that $f_L(k^2) = 0$, and we thus conclude that $\Pi^{\mu\nu}$ is purely transverse. That is, from (19), we have that

$$\Pi^{\mu\nu} = \left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) f_T(k^2). \tag{35}$$

For $\Pi^{\mu\nu}(k)$ to be non-singular at $k = 0$, we must have

$$f_T(k^2) = k^2 \Pi(k^2), \tag{36}$$

where $\Pi(0)$ is non-singular. Hence, for the two-point function in the interacting theory, we have

$$D_{\mu\nu} = \frac{-i}{k^2 - i\epsilon}\left[\frac{P^T_{\mu\nu}}{1 - \Pi(k^2)} + \xi P^L_{\mu\nu}\right]. \tag{37}$$

Remarks:

1. The longitudinal part of $D_{\mu\nu}$ does not receive any loop corrections: it is completely determined by the Ward identities. The physics should not depend on this part. For example, in the Landau gauge, $\xi = 0$, $D_{\mu\nu}$ is purely transverse.

2. Since $\Pi(k^2)$ is non-singular at $k^2 = 0$, the photon remains massless to all orders. There are exceptions to this: it is not true in quantum electrodynamics in $1+1$ dimensions, or in theories where an additional Higgs field is introduced.

3. The residue at the $k^2 = 0$ pole is given by $Z_3^{-1} = 1 - \Pi(0)$, and we have that

$$iD^T_{\mu\nu} \approx \frac{Z_3}{k^2 - i\epsilon} P^T_{\mu\nu} \tag{38}$$

near $k^2 = 0$. The renormalized field is given by $A^B_\mu = \sqrt{Z_3} A_\mu$.