Quantum Field Theory II (8.324) Fall 2010
Assignment 7

Readings

- Peskin & Schroeder chapters 10, 12, 13.
- Weinberg vol 1 chapter 12 and Vol 2 chapter 18.

Problem Set 7

1. **Renormalization group properties (30 points)**

   (a) Consider a coupling constant λ and a redefined coupling constant $\bar{\lambda}(\lambda)$. Find the general transformation law for the beta function, namely the relation between $\beta(\lambda)$ and $\overline{\beta}(\overline{\lambda})$. If we think of λ as a coordinate we see that β transforms as a tensor. What kind of tensor ?

   (b) Assume that
   $$\beta(\lambda) = b_2\lambda^2 + b_3\lambda^3 + b_4\lambda^4 + \cdots$$
   and consider the perturbatively defined and invertible coupling constant redefinition:
   $$\bar{\lambda}(\lambda) = \lambda + a_2\lambda^2 + a_3\lambda^3 + \cdots .$$
   Calculate $\bar{\beta}(\bar{\lambda})$ writing it in the form
   $$\bar{\beta}(\bar{\lambda}) = \bar{b}_2\bar{\lambda}^2 + \bar{b}_3\bar{\lambda}^3 + \bar{b}_4\bar{\lambda}^4 + \cdots$$
   Verify that:

   i. $\bar{b}_2 = b_2$ and $\bar{b}_3 = b_3$.

   ii. It is possible to make $\bar{b}_4$ anything you want by such a coupling redefinition.

   iii. Let $\lambda = \lambda_F$ denote a fixed point. Show that $\bar{\lambda} = \bar{\lambda}_F$ is also a fixed point. How are the derivatives β' and $\bar{\beta}'$ related at the fixed point?

(c) Consider the differential equation for a massless coupling g

$$\mu \frac{dg}{d\mu} = -bg^2 - cg^3 - dg^4 - \cdots \tag{1}$$

with b, c, d numerical constants. Show that one can write a solution to the above equation in the form

$$\ln \frac{\mu}{\Lambda} = \frac{1}{bg(\mu)} + \frac{c}{b^2} \ln bg(\mu) + \mathcal{O}(g(\mu)) \tag{2}$$

where Λ is an integration constant, which can be considered as a dynamically generated scale. Argue that Λ is renormalization group invariant.

(d) More generally, show that in a renormalizable theory with a dimensionless coupling constant $g(\mu)$ and no other dimensional parameter (like in a non-Abelian gauge theory), dimensional transmutation happens. That is, show that $g(\mu)$ can be written in a form

$$g(\mu) = f\left(\log\left(\frac{\mu}{\Lambda}\right)\right) \tag{3}$$

with Λ a universal scale and f a function depending on the specific theory. (Inverting (2) gives a specific example of f.)

2. **Asymptotic freedom in six-dimensional field theory (30 points)**

Consider the field theory

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{g_0}{6}\phi^3 \tag{4}$$

Calculate the β-function associated with the coupling g_0. Use dimensional regularization and minimal subtraction. You will find that this is a theory where the coupling constant becomes weaker at higher energies *i.e.* it is an asymptotically free theory.

3. **Asymptotic symmetry (40 points)**

Peskin & Schroeder prob. 12.3. (In intermediate steps you can use results of section 10.2 of Peskin & Schroeder.)

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