Quantum Field Theory II (8.324) Fall 2010
Assignment 2

Readings

- Prof. Zwiebach's notes on Lie algebras
- Peskin & Schroeder chapters 15 and 16
- Weinberg vol 2 chapter 15.
- After finishing non-Abelian gauge theories we will go back to Peskin & Schroeder chapter 6 and Weinberg Vol I chapter 11.

Note:

- In my lectures I will not discuss Batalin-Vilkovisky formalism. It is a beautiful formalism with many applications. It is particularly powerful in dealing with theories with complicated gauge symmetries for which ghosts for ghosts are needed. For example, this formalism played a crucial role in Prof. Zwiebach's seminal work on constructing a complete covariant closed string field theory. Those of you who would like to get a brief introduction about this formalism can read Weinberg's Vol 2 Sec.15.8–15.9.

Problem Set 2

1. **Group theory (14 points)**

   (a) Peskin & Schroeder prob. 15.1 (c) and (d) only

   (b) Peskin & Schroeder prob. 15.2

2. **Transformation of the covariant current (12 points)**

Consider the usual Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - i\bar{\Psi}(\gamma^\mu D_\mu - m)\Psi \tag{1}$$

Here the unitary matrix group G has Lie algebra generators satisfying

$$[T_a, T_b] = i f_{abc} T_c \tag{2}$$

with f_{abc} totally antisymmetric. The matter field Ψ transforms in some Lie-algebra representation defined by matrices T^r_a. As we derived in lecture the equation of motion for the gauge field can be written as

$$D_\mu F^{\mu\nu} = J^\nu \tag{3}$$

with (in the expression below a-index is summed over)

$$F^{\mu\nu} = F^{\mu\nu a}T^r_a, \qquad J^\nu = g(\bar{\Psi}\gamma^\nu T^r_a \Psi)T^r_a \tag{4}$$

(a) Using the equations of motion prove that J^μ is covariantly conserved.

(b) Derive the transformation property of J^ν under a gauge transformation. and use this to show that equation (3) is consistent under a gauge transformation (i.e. both sides transform the same). Note that for this problem you **cannot** use the transformation property of the LHS of (3) to deduce that for J^μ as you are asked to checked the consistency of the equation.

3. θ **term (12 points)**

One can in principle add the following term to a Yang-Mills Lagrangian

$$\mathcal{L}_\theta = \frac{\theta}{64\pi^2}\epsilon^{\mu\nu\lambda\rho}F^a_{\mu\nu}F^a_{\lambda\rho} \tag{5}$$

(a) Discuss how this term transforms under C, P, T transformations.

(b) Express it explicitly as a total derivative.

Note: You will learn later (probably next term) that θ is in fact an angle with period 2π, i.e. with the particular normalization in (5), the theory is equivalent for $\theta + 2\pi n$ for any integer n.

4. **Flat connections are locally trivial (24 points)**

Consider a non-Abelian gauge theory based on a unitary matrix group and assume we have a configuration for the gauge field A_μ such that the corresponding field strength $F_{\mu\nu}$ vanishes. Such gauge field, or connection, is said to be a *flat connection*. We want to show that such a flat connection is gauge equivalent to zero locally; i.e., we can find a well-defined gauge parameter such that after a gauge transformation the gauge field can be made to vanish on a connected region of spacetime. Show this in the following way:

(a) Write a differential equation for the group element $U(x)$ that defines the gauge transformation that gauges away A_μ. Verify that the local integrability condition $[\partial_\mu, \partial_\nu]U(x) = 0$ is satisfied for the present problem.

(b) Choose an origin x_0 and take $U(x_0) = 1$. Consider a path $x(s)$ starting from $x_0 = x(s = 0)$. Write an ansatz for $U(x(s))$. To verify that U solves the differential equation requires consideration of more than one path. Explain why. Conclude that your ansatz solves the problem if the value of U at any point is independent of the path chosen.

(c) Show that $U(x)$, constructed by using paths, is in fact path independent (this is somewhat challenging!). Hints: Consider two nearby paths $x^\mu(s)$ and $\tilde{x}^\mu(s) = x^\mu(s) + \delta x^\mu(s)$, both of which start at x_0 and end at x_1. Let $U(x(s))$ and $\tilde{U}(\tilde{x}(s))$ denote the solutions along the paths and $\delta U = \tilde{U}(s) - U(s)$. Study the differential equation satisfied by $U^{-1}\delta U$. (This part derives a special case of the so-called non-Abelian Stokes theorem. Prof. Goldstone was one of the early people who derived it.)

5. **Non-abelian gauge fields are not fully specified by $F_{\mu\nu}$ (20 points)**

We learned that a connection A_μ whose field strength $F_{\mu\nu}$ vanishes is locally gauge equivalent to the zero connection. We now want to ask another question. Suppose two different connections have the same field strength, are the two connections gauge equivalent locally? In other words, is there a gauge transformation that maps one connection into the other (locally)?

(a) Consider the question in the Abelian theory and conclude that in this case the gauge fields are gauge equivalent locally.

For the non-abelian theory connections with the same field strength need not be gauge equivalent locally. Therefore, it is sufficient to show an example where this is the case. Here is an example from Prof. Jackiw. Consider an $SU(2)$ gauge theory and two connections. The first one lives in the third direction of the Lie algebra, and the only nonvanishing components are

$$A_x = -\frac{1}{2} g y \frac{\sigma^3}{2}, \quad A_y = \frac{1}{2} g x \frac{\sigma^3}{2} \tag{6}$$

where g is the gauge coupling constant and $\sigma^i, i = 1, 2, 3$ are Pauli matrices. The second gauge field will be of the form

$$A_x = \frac{\sigma^1}{2}, \quad A_y = \frac{\sigma^2}{2} \tag{7}$$

with all other components equal to zero.

(b) Show by explicit computation that both connections lead to the same field strength $F_{\mu\nu}$.

(c) It remains to show that the two connections given above are not gauge equivalent. Prove that this is the case by picking a suitable gauge covariant local object and showing that while it vanishes for one connection it does not vanish for the other (why is this sufficient ?)

6. **Quantization of non-Abelian gauge theories (18 points)**

(a) Derive the propagator for the gauge field in the axial gauge

$$A_z^a = 0 \ . \tag{8}$$

(b) Consider the generalized gauge Coulomb gauge, i.e. taking the gauge fixing function as $f_a = \vec{\nabla}\cdot\vec{A}_a$. Derive the ghost Lagrangian and ghost propagator.

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