Quantum Field Theory II (8.324) Fall 2010
Assignment 1

Readings

- Peskin & Schroeder chapters 15 and 16
- Weinberg vol 2 chapter 15.
- Prof. Zwiebach's notes on Lie algebras

Note:

- In lectures I will focus on presenting physical ideas and will not have time to introduce mathematical background extensively. For relevant background on Lie algebras and their representations please read Prof. Zwiebach's notes I posted on the web and Peskin & Schroeder's section 15.4 "Basic facts about Lie Algebras".
- There are deep connections between the geometric structure of Yang-Mills theory and Einstein's general relativity. For those of you who have studied general relativity should read the end of sec. 15.1 and sec. 15.3 of Weinberg Volume II.
- There is a rich mathematical structure behind Yang-Mills theory carrying the name fibre bundle. Those of you who are interested in digging into this a little deeper can find a nice description in the book by John Baez and Javier Muniain, "Gauge fields, knots and gravity", World Scientific (1994).
- Before the Yang-Mills paper in 1954, a few people came very close to the discovery of Yang-Mills theory, including Klein and Pauli. For a prehistory of Yang-Mills theory, see O'Raifeartaigh and Straumann, "Early history of gauge theories and Kaluza-Klein theories", available on-line at

  http://arxiv.org/abs/hep-ph/9810524.

  This paper also review attempts to unify gauge theories and general relativity.

  At almost the same time of the paper of Yang-Mills the theory was also discovered independently by Ron Shaw who wrote it in his Cambridge University Ph.D thesis, but never published it. Shaw later became a Mathematician at Hull University, UK. Some of you might find it interesting to read about Shaw's discovery in the reminiscence of Shaw himself:

  http://www.hull.ac.uk/php/masrs/reminiscences.html

Problem Set 1

1. **Non-Abelian global symmetries and the associated charges (30 points)**

   Consider the following Lagrangian

$$\mathcal{L} = -i\bar{\Psi}(\gamma^\mu \partial_\mu - m)\Psi \tag{1}$$

   where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \bar{\Psi} = \left(\bar{\psi}_1, \bar{\psi}_2\right) \tag{2}$$

   $\psi_{1,2}$ are Dirac spinor fields. We will suppress spinor indices throughout.

   (a) Show that (1) is invariant under infinitesimal transformations

$$\delta\Psi = i\epsilon_a T_a \Psi, \qquad T_a = \frac{\sigma_a}{2}, \quad a = 1, 2, 3 \tag{3}$$

   where σ_a are Pauli matrices.

   (b) Find the conserved currents J_a^μ corresponding to the symmetric transformations (3).

   (c) Write down the corresponding conserved charges $Q_a, a = 1, 2, 3$. Show that

$$\delta\Psi = i[\epsilon_a Q_a, \Psi] \tag{4}$$

   (d) Find the commutation relations between Q_a's (using the canonical commutators).

   (e) Let

$$\hat{U} = \exp\left(i\Lambda_a Q_a\right), \qquad U = \exp\left(i\Lambda_a T_a\right) \tag{5}$$

   for some constants Λ_a. Note that $\hat{U}$ is a quantum operator, while U is a 2×2 unitary matrix. Show that

$$\hat{U}\,\Psi\,\hat{U}^\dagger = U\Psi\ . \tag{6}$$

2. **Parallel transport around a small loop (20 points)**

   Consider a *small* closed loop C which we take to be a parallelogram with one corner at x^μ and two sides a^μ and b^μ. Denote the transport around the loop to be $U_C(x, x)$. Show that $U_C(x, x)$ can be expressed in terms of the field strength in both $U(1)$ and general non-Abelian case. You should write down the explicit expressions of $U_C(x, x)$ in terms of the field strength. State how this result can be generalized to an arbitrary *small* closed loop.

3. **Bianchi identity (15 points)**

Check the Bianchi identity

$$D_\mu F_{\nu\lambda} + D_\lambda F_{\mu\nu} + D_\nu F_{\lambda\mu} = 0 \tag{7}$$

where

$$D_\mu F_{\nu\lambda} \equiv \partial_\mu F_{\nu\lambda} - ig[A_\mu, F_{\nu\lambda}] \tag{8}$$

4. **Scalar propagator in a gauge theory (35 points)**

   (a) Peskin & Schroeder prob. 15.4 part (a)

   (b) Peskin & Schroeder prob. 15.4 part (b)

   (c) Consider n complex scalar fields of mass m arranged in a vector

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} \tag{9}$$

   Write down a $U(n)$ gauge invariant Lagrangian for Φ. You can assume that Φ has no self-interactions.

   (d) Peskin & Schroeder prob. 15.4 (c) (using the Lagrangian you obtained in part (c) above).

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