8.323 Relativistic Quantum Field Theory I
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

Prof. Alan Guth — March 13, 2008

PROBLEM SET 5

REFERENCES: *Lecture Notes #4: Dirac Delta Function as a Distribution*, on the website. Peskin and Schroeder, Sec. 2.4; Optional reference: *Quantum Field Theory*, by Lowell Brown, section 1.7 (coherent states).

Problem 1: Subtleties of delta functions *(10 points)*

(a) Consider $g_1(t)$ and $g_2(t)$ defined by

$$g_1(t) \equiv f(t)\,\delta(t-a) \tag{1.1}$$

and

$$g_2(t) \equiv f(a)\,\delta(t-a)\ , \tag{1.2}$$

where $f(t)$ is an arbitrary smooth function, a is a constant, and δ denotes the Dirac delta function. By "smooth," I mean that $f(t)$ is continuous, and differentiable as many times as might be necessary for any issue that arises. Note that $g_1(t)$ and $g_2(t)$ are distributions, defined more explicitly as

$$T_{g_i}[\varphi] \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, g_i(t)\,\varphi(t)\ , \tag{1.3}$$

where $\varphi(t)$ is a test function. By evaluating the functionals shown in Eq. (1.3) for an arbitrary allowed test function, find out if $g_1(t)$ and $g_2(t)$ are equal to each other as distributions.

(b) Using a prime to indicate that a function is differentiated with respect to its argument, consider the distributions $h_1(t)$ and $h_2(t)$ defined by

$$h_1(t) \equiv f(t)\,\delta'(t-a) \tag{1.4}$$

and

$$h_2(t) \equiv f(a)\,\delta'(t-a)\ , \tag{1.5}$$

where again $f(t)$ is an arbitrary smooth function and δ denotes the Dirac delta function. The derivative of a distribution was defined in Lecture Notes 4. By evaluating these two distributions for an arbitrary allowed test function $\varphi(t)$, find out if they are equal to each other as distributions.

(c) Now consider the derivatives of the distributions defined above in Eqs. (1.1) and (1.2):

$$g_1'(t) = f'(t)\,\delta(t-a) + f(t)\delta'(t-a) \tag{1.6}$$

and

$$g_2'(t) = f(a)\delta'(t-a) \; . \tag{1.7}$$

<u>IF</u> you concluded that $g_1(t)$ and $g_2(t)$ are equal, then you should certainly expect that their derivatives should be equal, even if they do not appear to be identical. Find out if $g_1'(t)$ and $g_2'(t)$ are equal.

(d) Let the function $\theta(t)$ be defined in the standard way,

$$\theta(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{1.8}$$

and consider the corresponding distribution

$$T_\theta[\varphi] \equiv \int_{-\infty}^{\infty} \mathrm{d}t\,\theta(t)\,\varphi(t) \; . \tag{1.9}$$

Define the derivative of a θ-function as a distribution, so

$$\int_{-\infty}^{\infty} \mathrm{d}t\,\theta'(t)\,\varphi(t) \equiv T_\theta'[\varphi] \equiv -T_\theta\left[\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right] \; . \tag{1.10}$$

Show that

$$\theta'(t) = \delta(t) \; . \tag{1.11}$$

[**Warning:** The product of two distributions cannot be defined in general, nor can the square of a distribution. As a function, it is clear from the definition (1.8) that

$$\theta^n(t) = \theta(t) \; , \tag{1.12}$$

where n denotes any positive integer. Any function that is piecewise continuous and bounded by a power can be promoted to a corresponding distribution, so $T_\theta[\varphi]$ and $T_{\theta^n}[\varphi]$ are both well-defined distributions and are equal to each other. One might conjecture that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\theta(t) = \delta(t) \; , \tag{1.13a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\theta^2(t) \underset{?}{=} 2\,\theta(t)\,\delta(t) \; , \tag{1.13b}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\theta^3(t) \underset{?}{=} 3\,\theta^2(t)\,\delta(t) \ . \tag{1.13c}$$

Eq. (1.13a) is identical to Eq. (1.11) and is correct, but the right-hand sides of Eqs. (1.13b) and (1.13c) are ill-defined. Since $\theta^n(t) = \theta(t)$, the left-hand sides of Eqs. (1.13a) – (1.13c) are all well-defined and are equal to each other. It is hard to imagine any consistent definition that would make the right-hand sides equal, so the standard approach is to consider them undefined.]

Problem 2: $\left(\Box_x + m^2\right) D_F(x-y) = -i\delta^{(4)}(x-y)$ *(10 points)*

The Feynman propagator for a free scalar field $\phi(x)$ is defined by

$$D_F(x-y) = \theta(x^0 - y^0)\,\langle 0\,|\phi(x)\,\phi(y)|\,0\rangle + \theta(y^0 - x^0)\,\langle 0\,|\phi(y)\,\phi(x)|\,0\rangle \ ,$$

where $|0\rangle$ denotes the vacuum state. Use the canonical commutation relations to show that

$$\left(\Box_x + m^2\right) D_F(x-y) = -i\delta^{(4)}(x-y) \ ,$$

where

$$\Box_x \equiv \eta^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} \ .$$

[*Suggestion:* First calculate

$$\frac{\partial}{\partial x^0} D_F(x-y) \ ,$$

and then differentiate this expression again to find

$$\frac{\partial^2}{\partial\left(x^0\right)^2} D_F(x-y) \ .$$

Then add in the other terms to find the final answer.]

Problem 3: Coherent states *(15 points)*

In lecture we solved the problem of a quantized scalar field $\phi(x)$ interacting with a fixed classical source $j(x)$,

$$(\Box + m^2)\phi(x) = j(x) \ .$$

We found that the in and out operators are related by

$$\phi_{\text{out}}(x) = S^{-1}\,\phi_{\text{in}}(x)\,S$$
$$a_{\text{out}}(\vec{p}) = S^{-1}\,a_{\text{in}}(\vec{p})\,S \ ,$$

where S can be written

$$S = e^{-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\tilde{\jmath}(p)|^2}\, e^F\, e^G \ ,$$

where

$$F = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(p)\, a^\dagger_{\text{in}}(\vec{p})$$

$$G = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(-p)\, a_{\text{in}}(\vec{p}) \ ,$$

and

$$\tilde{\jmath}(p) \equiv \int d^4y\, e^{ip\cdot y}\, j(y) \ .$$

(a) Show that S can also be written as

$$S = e^{-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\tilde{\jmath}(p)|^2}\, e^{F'}\, e^{G'} \ ,$$

where

$$F' = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(p)\, a^\dagger_{\text{out}}(\vec{p})$$

$$G' = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\tilde{\jmath}(-p)\, a_{\text{out}}(\vec{p}) \ ,$$

and hence that

$$|0_{\text{in}}\rangle = e^{-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\tilde{\jmath}(p)|^2}\, e^{F'}\, |0_{\text{out}}\rangle \ .$$

Note that the right-hand-side of the above equation gives a useful description of the final state, since the out operators have a straightforward interpretation at late times. States of this form — exponentials of creation operators acting on the vacuum — are called coherent states.

(b) To study further the properties of coherent states, it is useful to consider a single harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{\omega^2 m}{2} q^2 \ ,$$

which is simplified by the canonical transformation

$$p = \sqrt{m\omega}\,\bar{p} \ , \quad q = \frac{1}{\sqrt{m\omega}}\,\bar{q} \ .$$

Dropping the overbars, the creation and annihilation operators are then given by

$$a^\dagger = \frac{1}{\sqrt{2}}(q - ip)$$
$$a = \frac{1}{\sqrt{2}}(q + ip) \ .$$

A coherent state $|z\rangle$ can be defined by

$$|z\rangle \equiv e^{za^\dagger} |0\rangle \ .$$

Show that $|z\rangle$ is an eigenstate of the annihilation operator, and find its eigenvalue.

(c) Show that $\langle z_2 | z_1 \rangle = e^{z_2^* z_1}$. (If you look at Lowell Brown's book, note that I am defining $\langle z|$ to be the bra vector that corresponds to the ket $|z\rangle$, so my $\langle z|$ is equal to Brown's $\langle z^*|$.)

(d) Find

$$\langle q \rangle_z \equiv \frac{\langle z | q | z \rangle}{\langle z | z \rangle}$$

and

$$\langle p \rangle_z \equiv \frac{\langle z | p | z \rangle}{\langle z | z \rangle} \ .$$

(e) Compute the standard deviations of q and p,

$$\Delta q^2 = \left\langle (q - \langle q \rangle)^2 \right\rangle$$
$$\Delta p^2 = \left\langle (p - \langle p \rangle)^2 \right\rangle$$

and show that $|z\rangle$ is a minimal-uncertainty state, in the sense that

$$\Delta q \, \Delta p = \frac{1}{2} \ .$$