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8.323 Relativistic Quantum Field Theory I
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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

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PROBLEM SET 4

REFERENCES: *Lecture Notes #3: Distributions and the Fourier Transform*, on the website.

Problem 1: Evaluation of $\langle 0|\phi(x)\phi(y)|0\rangle$ for spacelike separations (10 points)

Problem 2.3 of Peskin and Schroeder. Alternatively — and this alternative might be more useful to your education — you can evaluate the expectation value by numerical integration, and draw a graph of the result. If you scale your axes with the appropriate powers of the mass m , the same graph can be valid for all values of m .

Problem 2: A tale of three cutoffs (15 points)

Consider the function

$$g(t) = \theta(t)e^{-i\omega_0 t} . \quad (1.1)$$

In this problem we will examine the Fourier transform of this function, which is defined formally by the ill-defined integral

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} g(t) . \quad (1.2)$$

While this integral does not exist, it should be possible to define the Fourier transform of $g(t)$ in the sense of distributions.

The Fourier transform will then not be a function, but instead will be a distribution, which we will denote generically as $F[\varphi]$, where $\varphi(t)$ is the test function on which the distribution will act. The function $g(t)$ itself is replaced by the distribution

$$F_g[\varphi] \equiv \int_{-\infty}^{\infty} dt g(t) \varphi(t) . \quad (1.3)$$

- (a) Let $F^{(1)}[\varphi]$ denote the Fourier transform of $F_g[\varphi]$, in the formal sense of distributions, which can also be denoted by $\tilde{F}_g[\varphi]$. Evaluate $F^{(1)}[\varphi]$ for the specific choice

$$\varphi(\omega) = \varphi_0(\omega) \equiv \frac{1}{\sqrt{2\pi}\sigma} e^{-(\omega-\omega_1)^2/(2\sigma^2)} . \quad (1.4)$$

You may leave your answer in the form of a definite integral, or you can evaluate it explicitly in terms of the error function (also called the Fresnel integral)

$$\Phi(x) \equiv \operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2} . \quad (1.5)$$

- (b) Another way to give meaning to expression (1.2) is to insert a convergence factor, replacing the original function by

$$g_\epsilon(t) = \theta(t) e^{-i\omega_0 t} e^{-\epsilon t} . \quad (1.6)$$

This function can be Fourier-transformed by the usual definition,

$$\tilde{g}_\epsilon(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} g_\epsilon(t) . \quad (1.7)$$

As a function there is no way to compare this with the answer in part (a), but we can promote it to a distribution by defining

$$F^{(2)}[\varphi] \equiv \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} d\omega \tilde{g}_\epsilon(\omega) \varphi(\omega) . \quad (1.8)$$

Evaluate this functional for $\varphi_0(\omega)$ given by Eq. (1.4). [*Hint*: you may want to make use of the fact that if an integral is absolutely convergent, you can exchange orders of integration without worry.]

- (c) The distribution theory approach guarantees us that we can use any cutoff function that we want, provided only that

$$|g_\epsilon(t)| < g(t) \quad \text{for } \epsilon > 0 \quad (1.9a)$$

and

$$\lim_{\epsilon \rightarrow 0} g_\epsilon(t) = g(t) \quad \text{for each } t, \quad (1.9b)$$

where the limit is not required to be uniform in t . Another possible cutoff, therefore, would be to simply truncate the integration at $\Lambda = 1/\epsilon$. So let

$$g_\epsilon^{(3)}(t) = \theta(t) e^{-i\omega_0 t} \theta\left(\frac{1}{\epsilon} - t\right) . \quad (1.10)$$

This function can also be Fourier-transformed, in analogy to Eq. (1.7), but the result $\tilde{g}_\epsilon^{(3)}(\omega)$ looks very little like $\tilde{g}_\epsilon(\omega)$. However, we can define the distribution

$$F^{(3)}[\varphi] \equiv \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} d\omega \tilde{g}_\epsilon^{(3)}(\omega) \varphi(\omega) . \quad (1.11)$$

Evaluate this functional for $\varphi_0(\omega)$ given by Eq. (1.4). If all goes well you should be able to show that it is equal to the previous two cases.

- (d) Small violations of the condition (1.9a) do not usually matter, but if one violates it grossly one can indeed construct cutoff functions $g_\epsilon(t)$ which are still consistent with condition (1.9b), but which lead to distributions that are not equivalent to $F^{(1)}[\varphi]$. Construct such a cutoff function. You need not evaluate $F[\varphi_0]$ for your case, but carry it far enough to show that the answer is different from the previous cases.