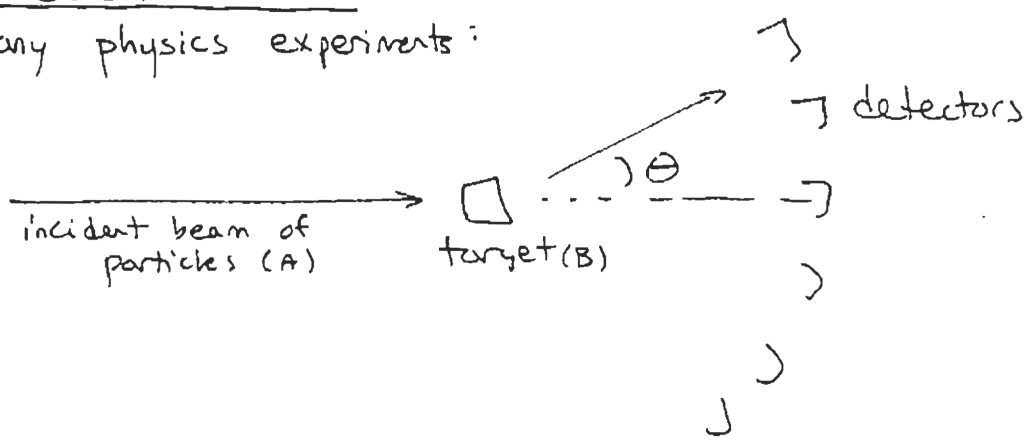
8. Scattering

8.1 General discussion

Many physics experiments:



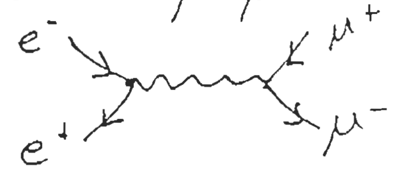
possible outcomes:

scattering:

1) elastic scattering:
   particle A bounces off target w/out changing internal structure of A or B
   (ex. $e^- - e^-$ scattering)

2) inelastic scattering:
   internal structure of A or B changes;
   kinetic energy absorbed in collision
   (ex. e^- scattering off H atom, collision raises energy of H)

3) rearrangement collisions:
   outgoing particles are not A & B
   (ex. $e^- e^+ \to \mu^- \mu^+$



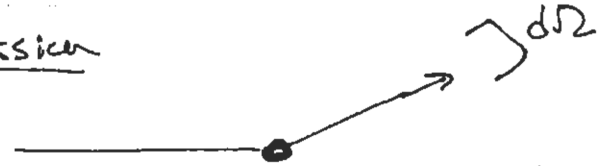
)

We will focus on elastic scattering

simplifying assumptions

- particles A, B spinless
- assume A, B have no internal structure (elastic)
- assume interaction from potential $V(\vec{r}_1 - \vec{r}_2)$
  - in COM frame, scattering of a particle of mass μ by potential $V(\vec{x})$ (potential scattering)

General discussion



(θ, ϕ)

F = flux

number of particles per unit angle: $d\sigma$

$$dn/dt = F \overbrace{\sigma(\theta,\phi)\, d\Omega}^{d\sigma}$$

↑ differential scattering x-section

$$\sigma = \int \sigma(\theta,\phi)\, d\Omega \quad : \quad \text{total scattering x-section}$$

Units: area (1 barn $= 10^{-24}$ cm^2)

$\sigma \sim$ cross sectional area of target

Looking for solution to

$$\left(\frac{p^2}{2m} + V\right)|\psi\rangle = E|\psi\rangle$$

$$\underbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right)}_{H_0} \psi(x) = \underbrace{\frac{\hbar^2 k^2}{2m}}_{E} \psi(x)$$

solution has general form (choose $\hat{k} = \hat{z}$ for now)

$$\psi(x) = \frac{1}{(2\pi\hbar)^{3/2}}\Big[\underbrace{e^{ikz}}_{\text{incoming wave}} + \underbrace{f(\theta,\phi)\, e^{ikr}/r}_{\text{outgoing wave}}\Big]$$

asymptotic form from $(\nabla^2 + k^2)\dfrac{e^{ikr}}{r} = 0$ [more: partial waves]

details of $f(\theta,\phi)$ depend on target (V)

From definitions, $d\sigma = |f(\theta,\phi)|^2$

8.2 Lippman - Schwinger

Want to solve

$$(H_0 + V)|\psi\rangle = E|\psi\rangle$$

write

$$|\psi\rangle = |\phi\rangle + |\chi\rangle$$

$$|\phi\rangle = \lim_{V\to 0} |\psi\rangle = |p\rangle \qquad (\vec{p} = \hbar k \hat{x})$$

So need

$$V|\phi\rangle + H_0|\chi\rangle + V|\chi\rangle = E|\chi\rangle$$

$$\Rightarrow (E - H_0)|\chi\rangle = V|\psi\rangle$$

$$\Rightarrow |\chi\rangle = \frac{1}{E-H_0} V|\psi\rangle$$

$$\Rightarrow \boxed{|\psi\rangle = |\phi\rangle + \frac{1}{E-H_0} V |\psi\rangle}$$

<u>Lippman - Schwinger</u>

Note: $\frac{1}{E-H_0}$ somewhat formal.

Unlike in time - indep. pert. theory, can't project out $|p\rangle$, because of cts. spectrum [unless put in cutoff]

Can deal with singularity by moving off real axis

$$\frac{1}{E-H_0} \longrightarrow \frac{1}{E-H_0 \pm i\varepsilon}$$

[see book for details]

Lippman - Schwinger leads to perturbative expansion

$$|\psi\rangle = |\phi\rangle + \frac{1}{E-H_0} V |\phi\rangle + \frac{1}{E-H_0} V \frac{1}{E-H_0} V |\phi\rangle$$

$$+ \ldots$$

Recall time - independent pert. theory — reduces to

$$|n^{(k)}\rangle = \frac{Q_n}{E_n^{(0)} - H_0} V |n^{(k-1)}\rangle$$

when no corrections to E; $E^{(k)} = 0$, $k > 0$.

projects out stny state $|n^{(0)}\rangle$

Same equation — could derive LS by putting in box, applying TI pert. thy.
$E^{(n)} \to 0$, spectrum $\to$ continuum

Back to integral equation

$$(H_0 + V)|\psi\rangle = E|\psi\rangle$$

define $V(\underline{x}) = \frac{\hbar^2}{2m} U(\underline{x})$

$$\Rightarrow \quad (\nabla^2 + k^2)\psi(\underline{x}) = U(\underline{x})\psi(\underline{x}) \qquad (*)$$

Standard technique: Green's functions

Assume $(\nabla^2 + k^2) G(\underline{x}, \underline{x}') = \delta^{(3)}(\underline{x} - \underline{x}')$

If $\varphi(x)$ satisfies homogeneous eqn

$$(\nabla^2 + k^2)\varphi(\underline{x}) = 0$$

then $\psi(\underline{x}) = \varphi(\underline{x}) + \int d^3\underline{x}'\, G(\underline{x} - \underline{x}') U(\underline{x}')\psi(\underline{x}')$

satisfies $(*)$

$$(\nabla^2 + k^2)\psi(\underline{x}) = (\nabla^2 + k^2)\int d^3\underline{x}'\, G(\underline{x} - \underline{x}') U(\underline{x}')\psi(\underline{x}')$$

$$= U(\underline{x})\psi(\underline{x}) \quad \checkmark$$

So need Green's function for $\nabla^2 + k^2$

$\rightarrow$ Given by $-\frac{1}{4\pi} e^{\pm ikr}/r$

[Derivation: book - uses $\pm i\varepsilon$ prescription, contour integrals]
[HW: check]

Since we want

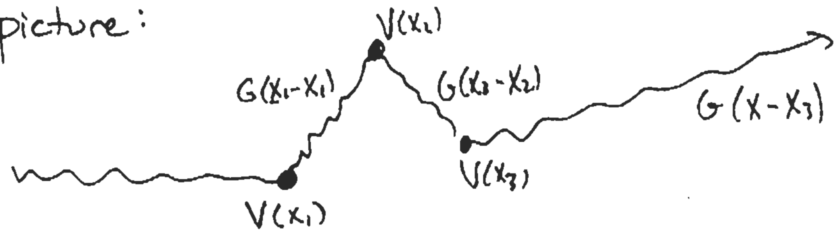
$$\psi(\underline{x}) = \frac{1}{(2\pi\hbar)^{3/2}}\left[e^{ikz} + f(\theta,\phi)\, e^{ikr}/r\right]$$

take G_+ for outgoing wave.

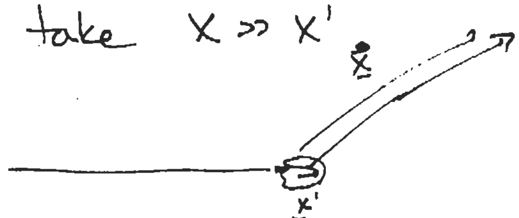
[G_- : outgoing plane wave ; difficult to arrange]

so $$\psi(\underline{x}) = \frac{1}{(2\pi\hbar)^{3/2}} e^{ikz} - \frac{2m}{\hbar^2}\int d^3\underline{x}'\, \frac{1}{4\pi}\, \frac{e^{ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|}\, V(\underline{x}')\,\psi(\underline{x}')$$

picture:



Scattering: take $x \gg x'$



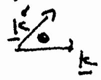
$$\frac{e^{ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} \sim \frac{e^{ikr}}{r}\, e^{i\underbrace{k\hat{\underline{x}}}_{\underline{k}'}\cdot\underline{x}'}$$

so asymptotic form of ψ:

$$\psi(\underline{x}) = \frac{1}{(2\pi\hbar)^{3/2}} e^{ikz} - \frac{2m}{4\pi\hbar^2} \frac{e^{ikr}}{r} \int d^3\underline{x}' e^{-i\underline{k}'\cdot\underline{x}'} V(\underline{x}')\psi(\underline{x}')$$

$$= \frac{1}{(2\pi\hbar)^{3/2}} \left[e^{ikz} + f(\theta,\phi)\, e^{ikr}/r \right]$$

$$\frac{1}{(2\pi\hbar)^{3/2}} f(\theta,\phi) = -\frac{2m}{4\pi\hbar^2} \int d^3\underline{x}'\, e^{-i\underline{k}'\cdot\underline{x}'} V(\underline{x}')\psi(\underline{x}')$$

[book: $f(\underline{k}', \underline{k})$ — same function, here just $\hat{k} = \hat{z}$.
generalize to any $\underline{k}, \underline{k}'$ — θ, ϕ are rel. angles ]
note: book drops $\hbar^{3/2}$ in state norm for $|p\rangle, |k\rangle$.

$$\frac{1}{(2\pi\hbar)^{3/2}} f(\underline{k}', \underline{k}) = -\frac{2m}{4\pi\hbar^2} \int d^3\underline{x}'\, e^{-i\underline{k}'\cdot\underline{x}'} V(\underline{x}')\psi(\underline{x}')$$

Integral equation difficult to solve; ψ on LHS, RHS

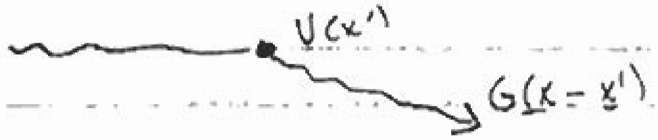
8.3 Born approximation

Use $|\psi\rangle = |p\rangle$ on RHS

$$f^{(1)}(\underline{k}, \underline{k}') = -\frac{2m}{4\pi\hbar^2} \int d^3\underline{x}'\, e^{i(\underline{k}-\underline{k}')\cdot\underline{x}'} V(\underline{x}')$$

gives
$$d\sigma^{(1)} = |f^{(1)}(\theta,\phi)|^2 d\Omega$$
$$= \frac{m^2}{4\pi^2\hbar^4} \left| \int d^3\underline{x}'\, e^{-i(\underline{k}-\underline{k}')\cdot\underline{x}'} V(\underline{x}') \right|^2$$

amounts to only taking one scattering event



<u>Born expansion</u>

1st order Born approx

$$\psi(\underline{x}) = \varphi(x) + \int d^3x'\, G(x-x')\, V(x')\, \varphi(x')$$
$$+ \int d^3x'\, d^3x''\, G(x-x')\, V(x')\, G(x'-x'')\, V(x'')\, \varphi(x'')$$

2nd Born approx

$$+ \cdots$$

Born approximation from TDPT (previously Lippman-Schwinger, TIPT)

Golden rule:
$$\omega_{p \to p'} = \frac{2\pi}{\hbar} \left| \langle p' | V | p \rangle \right|^2 \rho(E_k)$$

use box normalization $\quad |\underline{p}\rangle = \frac{1}{L^{3/2}} e^{i \underline{k} \cdot \underline{x}} \qquad k = \frac{\underline{p}}{\hbar} = \frac{2\pi \underline{n}}{L}$

$$\rho\, dE = n^2\, d\Omega\, dn = \left(\frac{L}{2\pi}\right)^3 \frac{km}{\hbar^2}\, dE\, d\Omega$$

$$\left(\frac{1}{d\Omega}\right) \omega_{p \to p'} = (\text{Incident flux}) \times \frac{d\sigma}{d\Omega}$$

$$\text{Flux} = |j| = \frac{\hbar}{m} \left| \operatorname{Im}\left(\frac{e^{-ikz}}{L^{3/2}} \vec{\nabla} \frac{e^{ikz}}{L^{3/2}} \right) \right| = \frac{\hbar k}{m L^3}$$

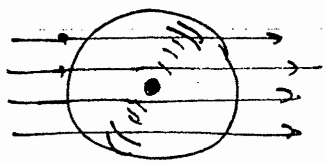
$$\text{so} \quad \frac{d\sigma}{d\Omega} = \frac{mL^3}{\hbar k}\cdot\frac{2\pi}{\hbar}\cdot\left(\frac{L}{2\pi}\right)^3\cdot\frac{km}{\hbar^2}\left|\frac{1}{L^3}\int d^3\underline{x}\, e^{i(\underline{k}-\underline{k}')\cdot\underline{x}}\, V(\underline{x})\right|^2$$

$$= \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3\underline{x}\, e^{i(\underline{k}-\underline{k}')\cdot\underline{x}}\, V(\underline{x})\right|^2 \quad \checkmark \qquad \text{(1st Born approx)}$$

8.4 Optical theorem

Consider conservation of probability flux $\underline{j} = \frac{\hbar}{m}\,\mathrm{Im}(\psi^* \underline{\nabla}\psi)$

Look at large sphere $r \to \infty$



net flux from plane wave $= 0$

net flux from scattering: $\sigma = \int |f|^2 d\Omega$

conserv. of prob: reduced amplitude @ $\theta = 0$ from interference

$$j_r = \frac{\hbar}{m}\,\mathrm{Im}\left(e^{-ikz} + f^*_{(\theta,\phi)}\frac{e^{-ikr}}{r}\right)\partial_r\left(e^{ikr\cos\theta} + f_{(\theta,\phi)}\frac{e^{ikr}}{r}\right)$$

(drop $\left(\frac{1}{2\pi\hbar}\right)^3$ or $\frac{1}{L^3}$)

$$= \frac{\hbar}{m}\Big[\underset{①}{k\cos\theta} + \underset{②}{|f|^2\frac{k}{r^2}}$$

$$+\,\mathrm{Im}\Big\{\frac{f^*}{r}\,ik\cos\theta\, e^{ikr(\cos\theta-1)} + \frac{f}{r}\,ik\, e^{ikr(1-\cos\theta)} - \frac{f}{r^2}\, e^{ikr(1-\cos\theta)}\Big\}_{③}\Big]$$

① flux from ~~incoming~~ plane wave

$$\int_{①} j_r \sim \int k\cos\theta\,\sin\theta\, d\theta = 0$$

② scattered flux

$$\int_{②} j_r = \int \frac{\hbar k}{m} |f|^2 \frac{1}{r^2} r^2 d\Omega = \frac{\hbar k}{m} \int |f|^2 d\Omega$$

$$= \frac{\hbar k}{m} \sigma_{tot} = (\text{Inc. flux}) \times \sigma$$

(L^3 .. $(2\pi)^3$ [illegible])

③ Interference term : oscillates rapidly (& cancels) as $r \to \infty$ unless $\theta = 0$.

Near $\theta = 0$,

$$\int_{③} j_r \cong \frac{\hbar}{m} 2\,\mathrm{Re} \int_{\theta=0} \frac{f(0)}{r} k\, e^{ikr(1-\cos\theta)} 2\pi r^2 \sin\theta\, d\theta$$

$$\cong \mathrm{Re}\, \frac{\hbar k}{m} 4\pi f(0) \int_{\theta=0}^{\infty} e^{ikr\left(\frac{\theta^2}{2}\right)} r\theta\, d\theta$$

$$\cong \mathrm{Re}\, \frac{4\pi \hbar k}{m} f(0) \lim_{\varepsilon \to 0} \int_{\theta=0}^{\infty} e^{ik(r+i\varepsilon)\theta^2/2} (r+i\varepsilon)\, \theta\, d\theta$$

$$= \frac{4\pi\hbar k}{m} \mathrm{Re}\left(f(0) \left[\frac{1}{ik} e^{ik(r+i\varepsilon)\theta^2/2} \right]_0^{\infty} \right)$$

$$= -\frac{4\pi\hbar}{m} \mathrm{Im}\, f(0).$$

Since total flux $= 0$ by prob. conserv.,

$$\boxed{\frac{4\pi}{k} \mathrm{Im}\, f(0) = \sigma_{tot}}$$ (Optical theorem)

Next: partial wave approach to scattering

Recall for decay of excited states

$|k\rangle$ basis is unnatural for photons w/ particular A.M.

– want vector spherical harmonics

Similar for scattering – use spherical waves

8.5 Spherical waves

Want to find (continuum) states of free particle w/ fixed A.M. eigenvalues L^2, L_z.

Fix $H = E = \frac{\hbar^2 k^2}{2m}$, $L^2 = \hbar^2 \ell(\ell+1)$, $L_z = \hbar m$

$$\psi(x) = \langle x | E, \ell, m \rangle = c\, R_\ell(r)\, Y_{\ell m}(\hat{x})$$

Free Schrödinger eqn:

$$R_\ell'' + \frac{2}{r} R_\ell' + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) R_\ell = 0$$

use $\rho = kr$

$$R_\ell''(\rho) + \frac{2}{\rho} R_\ell'(\rho) + R_\ell(\rho) - \frac{\ell(\ell+1)}{\rho^2} R_\ell(\rho) = 0$$

Solution: spherical Bessel functions

$$R_\ell(r) = a\, j_\ell(kr) + b\, n_\ell(kr)$$

$j_\ell(\rho) = \left(\frac{\pi}{2\rho}\right)^{1/2} J_{\ell+1/2}(\rho)$ Bessel fn (regular @ $\rho = 0$)

$n_\ell(\rho) = \left(\frac{\pi}{2\rho}\right)^{1/2} N_{\ell+1/2}(\rho)$ Neumann fn. (singular @ $\rho = 0$)

$$j_\ell(\rho) = (-\rho)^\ell \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \left(\frac{\sin\rho}{\rho}\right)$$
$$n_\ell(\rho) = -(-\rho)^\ell \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \left(\frac{\cos\rho}{\rho}\right)$$

$$j_0(\rho) = \frac{\sin\rho}{\rho} \qquad n_0(\rho) = \frac{-\cos\rho}{\rho}$$
$$j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho} \qquad n_1(\rho) = -\frac{\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho}$$
$$\vdots$$

small ρ:

$$j_\ell(\rho) \xrightarrow[\rho\to 0]{} \frac{\rho^\ell}{(2\ell+1)!!}$$
$$n_\ell(\rho) \xrightarrow[\rho\to 0]{} -\frac{(2\ell-1)!!}{\rho^{\ell+1}}$$

large ρ:

$$j_\ell(\rho) \xrightarrow[\rho\to\infty]{} \frac{1}{\rho}\sin\left(\rho - \frac{\ell\pi}{2}\right)$$
$$n_\ell(\rho) \xrightarrow[\rho\to\infty]{} -\frac{1}{\rho}\cos\left(\rho - \frac{\ell\pi}{2}\right)$$

Hankel functions

$$H^\pm_\ell(\rho) = H^{(1,2)}_\ell(\rho) = j_\ell(\rho) \pm i\, n_\ell(\rho) \xrightarrow[\rho\to\infty]{} (-i)^{\pm(\ell+1)} \frac{e^{\pm i\rho}}{\rho}$$

For functions $|E,\ell,m\rangle$ smooth @ origin, must have

$$\langle \underline{x} | E,\ell,m\rangle \sim j_\ell(kr)\, Y_{\ell m}(\hat{x})$$

Normalization of spherical wave:

want $\langle E', l', m' | E, l, m \rangle = \delta_{ll'} \delta_{mm'} \delta(E'-E)$

given by

$$\langle \underline{x} | E, l, m \rangle = \frac{i^l}{\hbar} \left(\frac{2mk}{\pi} \right)^{1/2} j_l(kr) \, Y_{lm}(\hat{x})$$

from $$\int_0^\infty r^2 dr \; j_l(kr) \, j_{l'}(k'r) = \frac{\pi}{2k^2} \delta(k-k')$$

[won't show]

Connect to p basis:

$$e^{ikz} = \sum_l a_l \, j_l(kr) \, P_l(\cos\theta)$$

since $\int_{-1}^{1} P_l(\xi) P_{l'}(\xi) \, d\xi = \frac{2}{2l+1} \delta_{l,l'}$

can use

$$\int_{-1}^{1} d\xi \, e^{ikr\xi} P_l(\xi) = \frac{2}{2l+1} a_l \, j_l(kr)$$

$$\overset{(HW)}{\Rightarrow} \quad e^{ikz} = \sum_l (2l+1) \, i^l j_l(kr) \, P_l\left(\frac{z}{r}\right)$$

more generally

$$e^{i\underline{k}\cdot\underline{x}} = \sum (2l+1) \, i^l j_l(kr) \, P_l(\hat{k}\cdot\hat{r})$$

$$\overset{(HW)}{\Rightarrow} \langle p | E, l, m \rangle = \frac{1}{\sqrt{mp}} \delta\left(E - \frac{p^2}{2m}\right) Y_{lm}(\hat{p})$$

8.6 Partial wave scattering.

Assume we have a potential $V(r)$
- spherically symmetric
- local $(V(r) = 0, \; r > R)$ [note: doesn't include Coulomb!]

Outside $r = R$, solutions are spherical waves $\psi = A_\ell(kr)\, Y_{\ell m}(\hat{x})$

$A_\ell(\rho) =$ linear combination of $j_\ell(\rho)$, $n_\ell(\rho)$

$h_\ell^{\pm}(\rho) \underset{\rho\to\infty}{\to} (\mp i)^{\pm(\ell+1)} \dfrac{e^{\pm i\rho}}{\rho}$

$= j_\ell(\rho) \pm i n_\ell(\rho)$

$h_\ell^+ =$ outgoing wave
$h_\ell^- =$ incoming wave

If $V = \emptyset$, lin comb. is $A_\ell = j_\ell$ — reg. at orig

For general potential, solution for $r > R$ is

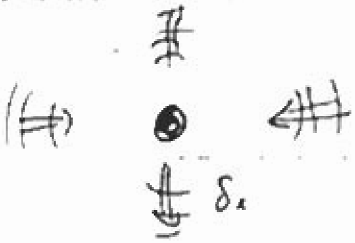
$$A_\ell(\rho) = \frac{C}{2}\left[h_\ell^-(\rho) + e^{2i\delta_\ell} h_\ell^+(\rho)\right]$$

Rotational symmetry $\Rightarrow$ ℓ, m incoming $= \ell, m$ outgoing (no mixing)

Unitarity (prob. conservation) $\Rightarrow$ $e^{2i\delta_\ell}$ pure phase (δ_ℓ real)

Scattering properties of V completely determined by phase shifts $e^{2i\delta_\ell}$ of partial waves.

Simple picture of scatterer



Determining phase shifts

Want to solve Schrödinger in potential $V(r)$

$$\psi_l(\underline{x}) = A_l(r)\, Y_{lm}(\hat{x})$$

with B.C. $A_l(0) = 0$.

Write $A_l(r) = \frac{1}{r} u_l(r)$

$$u_l'' + \left(k^2 - \frac{2m}{\hbar^2} V - \frac{l(l+1)}{r^2}\right) u_l = 0$$

Solve 1D Schrödinger out to $r = R$.

Then match $\beta_l = \dfrac{r}{A_l} \dfrac{dA_l(r)}{dr}$

with $\dfrac{c}{2}\left[h_l^-(\rho) + e^{2i\delta_l} h_l^+(\rho)\right]$

$$= c e^{i\delta_l}\left[\cos\delta_l\, j_l(\rho) - \sin\delta_l\, n_l(\rho)\right]$$

which has
$$\beta_l = kR\left[\frac{j_l'(kR)\cos\delta_l - n_l'(kR)\sin\delta_l}{j_l(kR)\cos\delta_l - n_l(kR)\sin\delta_l}\right]$$

$$\Rightarrow \quad \tan\delta_l = \frac{kR\, j_l'(kR) - \beta_l\, j_l(kR)}{kR\, n_l'(kR) - \beta_l\, n_l(kR)}$$

This allows us to fix the phase shifts δ_l for each l.

<u>Connection to plane wave scattering</u>

$$\psi(\underline{x}) \xrightarrow[r\to\infty]{} \frac{1}{(2\pi k)^{3/2}} \left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \right]$$

Define $f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(k) P_\ell(\cos\theta)$
[indep. of ϕ]

since $j_\ell(kr) \xrightarrow[r\to\infty]{} \dfrac{e^{i(kr-\ell\pi/2)} - e^{-i(kr-\ell\pi/2)}}{2ikr}$

& $e^{ikz} = \sum_\ell (2\ell+1)\, i^\ell\, j_\ell(kr) P_\ell(\cos\theta)$

$$\psi(\underline{x}) \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta) \frac{1}{2ik} \left[\underbrace{\frac{e^{ikr} - e^{-i(kr-\ell\pi)} + 2ik f_\ell e^{ikr}}{r}}_{[1+2ikf_\ell]\frac{e^{ikr}}{r} - \frac{e^{-i(kr-\ell\pi)}}{r}} \right]$$

matching with $\left(h_\ell^- + e^{2i\delta_\ell} h_\ell^+\right)$

$$\Rightarrow \boxed{e^{2i\delta_\ell} = 1 + 2ikf_\ell}$$

or $f_\ell = \dfrac{e^{2i\delta_\ell} - 1}{2ik} = \dfrac{e^{i\delta} \sin\delta_\ell}{k} = \dfrac{1}{k\cot\delta_\ell - ik}$

so $f(\theta) = \dfrac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell\, P_\ell(\cos\theta)$

Properties of phase shifts:

x– section

$$\sigma = \int |f(\theta)|^2 \, d\Omega$$

$$= \frac{1}{k^2} \cdot 2\pi \int_{-1}^{1} dz \sum_{l,l'} (2l+1)(2l'+1) e^{i(\delta_l - \delta_{l'})} \sin\delta_l \sin\delta_{l'} P_l(z) P_{l'}(z)$$

$$= \frac{4\pi}{k^2} \sum (2l+1) \sin^2 \delta_l$$

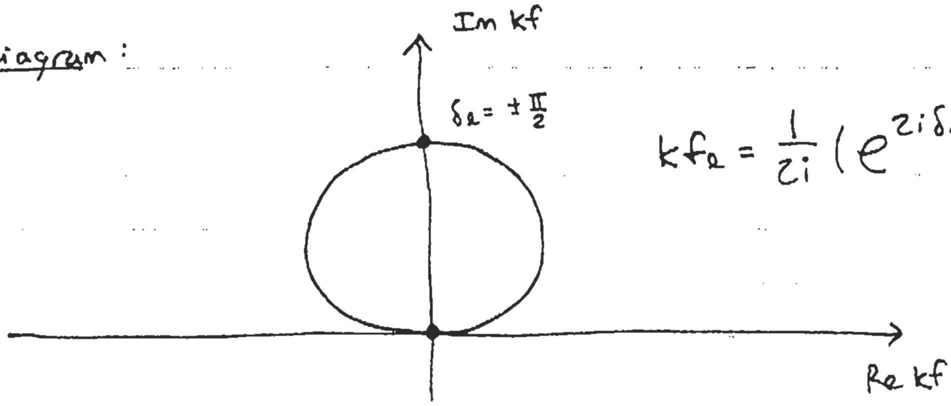
$$= \sum \sigma_l, \qquad \sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l$$

partial wave x-section

Optical theorem:

$$\operatorname{Im} f(0) = \sum \frac{(2l+1)}{k} \sin^2 \delta_l = \frac{k}{4\pi} \sigma_{tot} \qquad \checkmark$$

Argand diagram:



$$k f_l = \frac{1}{2i}\left(e^{2i\delta_l} - 1\right)$$

kf_l must lie on <u>unitarity circle</u>

$$\delta_l \text{ small} \Rightarrow f_l \cong \frac{\delta_l}{k} \quad \text{real}$$

Unitarity bound: $\sigma_l \leq \frac{4\pi}{k^2}(2l+1)$

Short-range potential:

If $V = 0$ except for $r \ll 1/k$

only $\ell = 0$ partial wave contributes ("s-wave")
since $j_\ell(kr) \sim 0$, $\ell \neq 0$ (incoming: only j_ℓ, no n_ℓ)

so $\sigma = \sigma_0 \leq \frac{4\pi}{k^2}$ no matter how strong potential is.

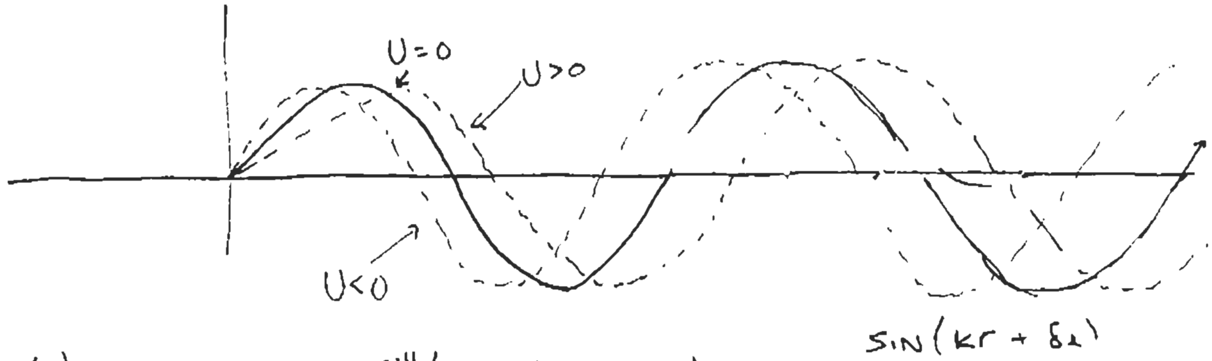
Similarly, for any k, R, @ large ℓ partial waves suppressed for $\ell \gg kR$ since $j_\ell \sim (kR)^\ell / (2\ell+1)!!$
(natural: no effect @ impact parameter $b = \frac{\ell}{k}$)

Attractive vs. repulsive potentials

Schrödinger : $\frac{u''}{u} = U + \frac{\ell(\ell+1)}{r^2} - k^2$

for large r, $u''/u < 0$.

$U < 0$ decreases u''/u (more negative)
- "pulls in" wfunction (attractive pot.)
$\Rightarrow \delta_\ell > 0$



$U > 0$ increases u''/u (less neg.)
- "pushes out" wfun.
$\Rightarrow \delta_\ell < 0$.

Example: Hard sphere

Consider
$$V(r) = \begin{cases} \infty, & r \leq R \\ 0, & r > R \end{cases}$$

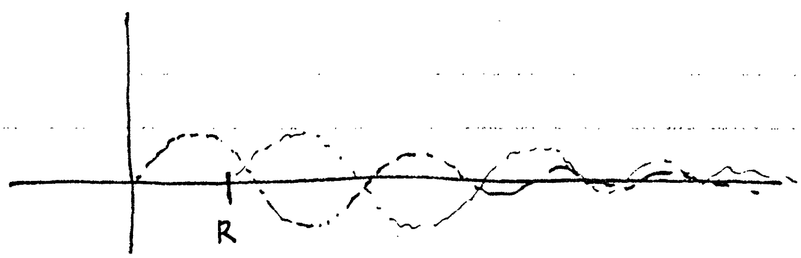
$$\Rightarrow \quad A_\ell(R) = 0$$

$$\Rightarrow \quad \cos\delta_\ell \, j_\ell(\rho) - \sin\delta_\ell \, n_\ell(\rho) \Big|_{\rho = kR} = 0$$

$$\Rightarrow \quad \tan\delta_\ell = \frac{j_\ell(\rho)}{n_\ell(\rho)}. \qquad \left[\Leftrightarrow \; e^{2i\delta} = -\frac{h_\ell^-(\rho)}{h_\ell^+(\rho)}\right]$$

s-wave:
$$\tan\delta_0 = -\tan kR$$

$$\Rightarrow \quad \delta_0 = -kR \qquad \text{(repulsive pot.)}$$



$$\psi_0 \sim \frac{1}{r}\sin(kr + \delta) = \frac{1}{r}\sin(k(r - R))$$

Low-energy limit

$$kR \ll 1$$

$$\tan\delta_\ell \sim \frac{\rho^\ell / (2\ell+1)!!}{-(2\ell-1)!! / \rho^{\ell+1}} = -(kR)^{2\ell+1} \frac{(2\ell+1)}{[(2\ell+1)!!]^2}$$

s-wave dominates.

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \longrightarrow \frac{1}{k^2}\sin^2\delta_0 \longrightarrow R^2$$

$$\sigma_{tot} = \int d\sigma = 4\pi R^2$$

(4× geometrical expectation; OK since λ large)

High-energy limit

Modes up to $l \sim kR$ contribute significantly
($j_l \sim (kR)^l / \underbrace{(2l+1)!!}_{\to l^l}$)

$$\sigma_{tot} = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2\delta_l$$

$$\sin^2\delta_l = \frac{\tan^2\delta_l}{1+\tan^2\delta_l} = \frac{j_l(kR)^2}{j_l(kR)^2 + n_l(kR)^2}$$

Asymptotically,

$$j_l(\rho),\ n_l(\rho) \longrightarrow \frac{1}{\rho}\sin\left(\rho - \frac{l\pi}{2}\right),\ -\frac{1}{\rho}\cos\left(\rho - \frac{l\pi}{2}\right)$$

asympt. behavior only accurate as $\rho \gg l$
but for $\rho \gtrsim l$, oscillating with ~ same amplitude.

so δ_l roughly randomly distributed phase as l goes from $0 \to \rho$.

[Sak. claim: $\sin^2\delta_l + \sin^2\delta_{l+1} \cong 1$ false for $l \lesssim \rho$. Such as $l = \rho/2$.]

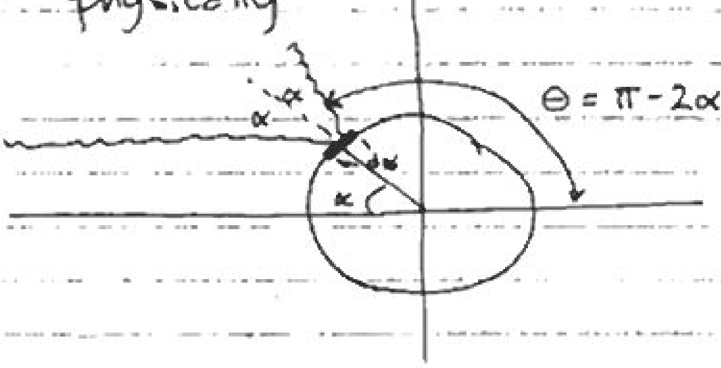
can still approx. for large kR: $\overline{\sin^2\delta_\ell} = 1/2$

$$\Rightarrow \sigma_{tot} \cong \frac{4\pi}{k^2}\sum_{\ell=0}^{kR}(2\ell+1)\cdot\frac{1}{2} \xrightarrow[kR\to\infty]{} 2\pi R^2$$

Still $2\times$ geometric x-section!

[note: answer correct though argument heuristic; arg. in Sak. wrong]

Expect physically



Incoming flux for $d\alpha\, d\phi = F\, R^2 \sin\alpha \cos\alpha\, d\phi\, d\alpha$

scattered flux $F\sigma(\theta,\phi)\sin\theta\, d\theta\, d\phi$

$$= F\sigma(\theta,\phi)\cdot 4\cdot\sin\alpha\cos\alpha\, d\alpha\, d\phi$$

$$\Rightarrow \sigma(\theta,\phi) = R^2/4 \quad \text{constant in } \theta.$$

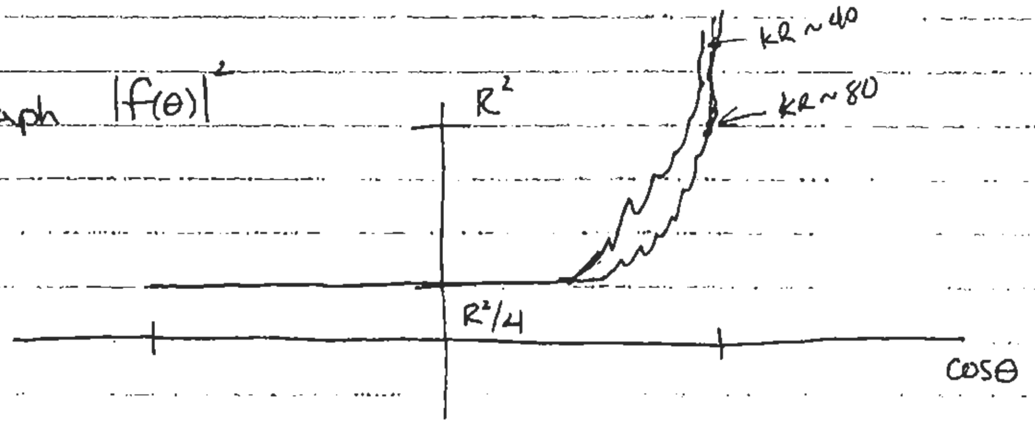
$$\sigma_{tot} = \int \sigma(\theta,\phi)\, d\Omega = \underline{\pi R^2}$$

why is this not what we get?

Use $f_\ell = \frac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right)$

$f(\theta) = \sum (2l+1)\, f_l\, P_l(\cos\theta)$

Graph $|f(\theta)|^2$



For fixed θ, as $kR \to \infty$, $f(\theta) \to 1/4$.

So get expected result.

"Extra" σ from "shadow" part of f.

$$f(\theta) = f_{refl} + f_{shadow}$$

$$\cong \sum_{l=0}^{kR} (2l+1) \frac{1}{2ik} e^{2i\delta_l} P_l(\cos\theta) + \sum_{l=0}^{kR} (2l+1) \frac{i}{2k} P_l(\cos\theta)$$

after $l \approx kR$, contributions cancel $(e^{2i\delta_l} \sim 1)$

Reflected x-section:

$$\int |f_{refl}|^2 d\Omega = \frac{2\pi}{4k^2} \sum_{l,l'=0}^{kR} (2l+1)(2l'+1) P_l(\cos\theta) P_{l'}(\cos\theta)$$

$$= \frac{\pi}{k^2} \sum_{l=0}^{kR} (2l+1) \cong \pi R^2$$

<u>Shadow x-section</u> :

$$f_{\text{shadow}} = \frac{i}{2k} \sum (2\ell+1) P_\ell(\cos\theta)$$

but $\frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta) \sim \delta(\cos\theta - 1)$

$$\left[\int d\xi \; \delta(\xi - 1) P_{\ell'}(\xi) = P_{\ell'}(1) = 1 \right.$$
$$\left. \int \frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\xi) P_{\ell'}(\xi)\, d\xi = 1 \right]$$

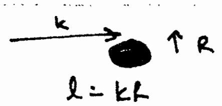
so f_{shadow} only relevant near $\theta = 0$.

represents part of f which cancels plane wave e^{ikz} in shadow of sphere.

Recall outgoing wave is $[1 + 2ik f_\ell] \frac{e^{ikr}}{r} (2\ell+1) P_\ell(\cos\theta)$

so $f_\ell = i/2k$ is what cancels plane wave.

Cancellation up to $\ell \sim kR$
affects impact parameter up to R.



Can check $\int |f_{\text{shadow}}|^2 d\Omega = \pi R^2$

$$\int f^*_{\text{shadow}} f_{\text{refl}} = 0.$$

(from oscillations of f_{refl}.)

8.7 Low-energy scattering, & bound states, & resonances

If $\quad kR \ll 1$

$$e^{2i\delta_\ell} \sim 1 \quad \text{for} \quad \ell \gg 1$$

more precisely

$$\tan \delta_\ell \cancel{\cong \frac{j_\ell(\rho)}{\sqrt{j_\ell^2(\rho) + n_\ell^2(\rho)}}} \cong \frac{j_\ell(\rho)}{n_\ell(\rho)} \cong \frac{\rho^{2\ell+1}}{(2\ell+1)!!(2\ell-1)!!}$$

$$\delta_\ell \sim (kR)^{2\ell+1}$$

S-wave most important contribution at low energy, $\delta_0 \sim kR$.

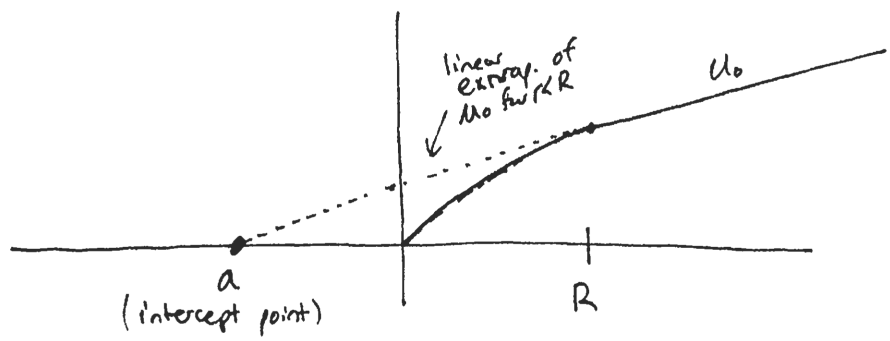
Consider outside wavefunction $(r > R)$ @ low energy

$$A_0 \propto [\cos\delta_0 \, j_0(kr) - \sin\delta_0 \, n_0(kr)]$$

$$u_0 = r A_0 \propto [\cos\delta_0 \sin kr + \sin\delta_0 \cos kr]$$

for $\quad k \ll \frac{1}{R}, \quad r > R \quad$ but $\quad kr \ll 1,$

$$u_0 \cong c \sin\delta_0 \, [1 + \cot\delta_0 \, kr]$$



$$a = \lim_{k\to 0} \frac{-1}{k\cot\delta_0}$$

so as $k \to 0 \quad \delta_0 \sim -ka$

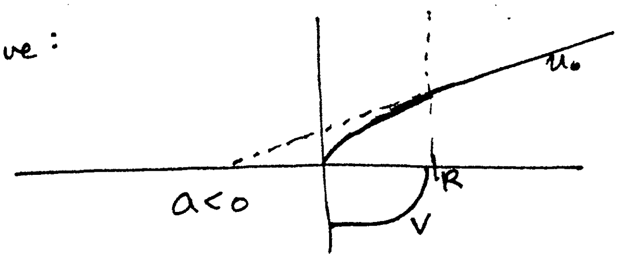
Relate to total x-section:

$$\lim_{k\to 0} \sigma_{tot} \Rightarrow \sigma_0 = 4\pi |f_0|^2 = 4\pi \left| \frac{1}{k\cot\delta_0 - ik} \right|^2$$

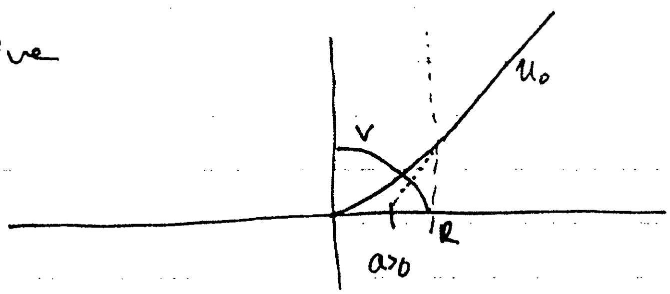
$$\Rightarrow 4\pi a^2 \qquad \text{(like hard sphere of radius } a\text{)}$$

a = "scattering length" gives limit of low-energy x-sectn.

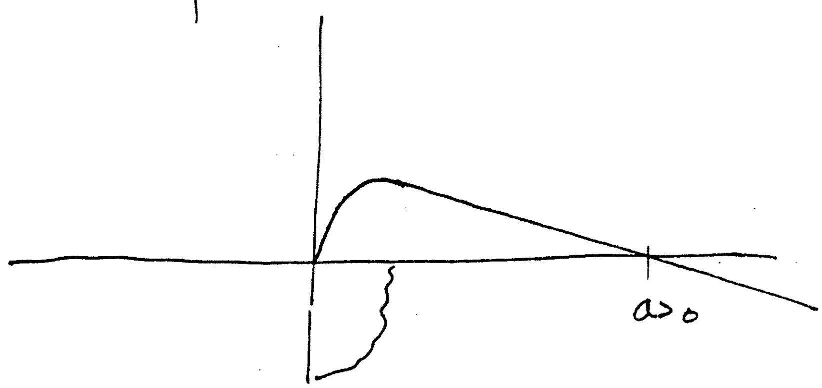
Attractive:



Repulsive



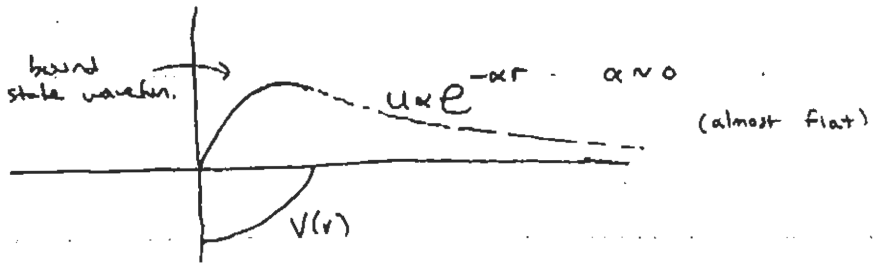
deeper attractive potential



change of sign signals bound state!

Scattering length & bound states

Suppose $V(r)$ has a barely bound state with $E_b = -\frac{\hbar^2\alpha^2}{2m} \lesssim 0$



u looks like inside wavefun of scattering state as $k \to 0$.

so $$\frac{r\frac{d}{dr} e^{-\alpha r}\Big|_{r=R}}{e^{-\alpha R}} = -\alpha \cong \lim_{k\to 0} \frac{r u_0'}{u_0} = \lim_{k\to 0} k\cot\delta_0 = -\frac{1}{a}$$

$$(u_0 \propto 1 + (\cot\delta_0)\, kr)$$

$$\Rightarrow \alpha \cong \frac{1}{a}$$

Binding energy $-E_b = \frac{\hbar^2\alpha^2}{2m} \cong \frac{\hbar^2}{2ma^2}$

so a large positive scattering length a
$\Longleftrightarrow$ weakly bound state $E \sim -\frac{\hbar^2}{2ma^2}$.

Bound states as poles

$l = 0$ wavefunction at large r

$$S_0(k)\frac{e^{ikr}}{r} - \frac{e^{-ikr}}{r} \qquad S_0(k) = e^{2i\delta_0}$$

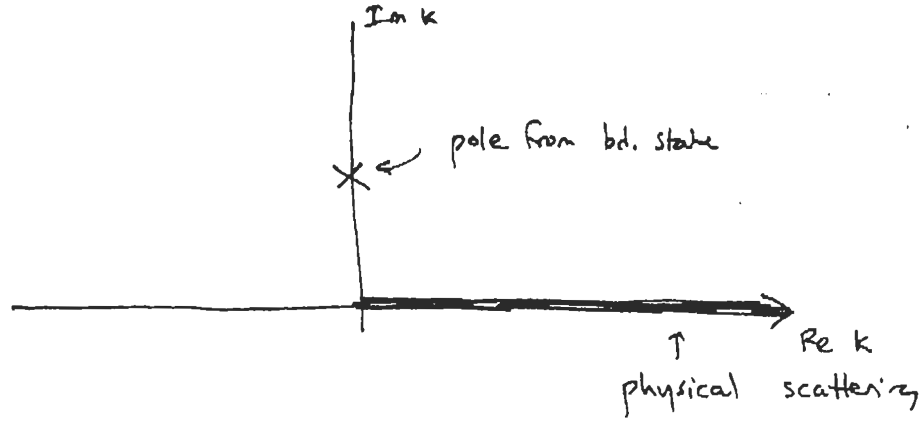
(outgoing) (incoming)

[Notation from S-matrix: outgoing = $\begin{pmatrix} S_0 & & \\ & S_1 & \\ & & \ddots \end{pmatrix}$ · incoming.
generalizes: $V(\vec{x})$ not rot. invariant: S not diagonal
[illegible]]

Bound state: asymptotically $e^{-\alpha r}/r$

Fits form above where $k = i\alpha$, $S_0(i\alpha) = \infty$
or $k = -i\alpha$, $S_0(-i\alpha) = 0$

$\Rightarrow$ Bound states appear as poles of $e^{2i\delta_0}$ & $kf_0 = \frac{1}{2i}(e^{2i\delta_0} - 1)$ on Im axis



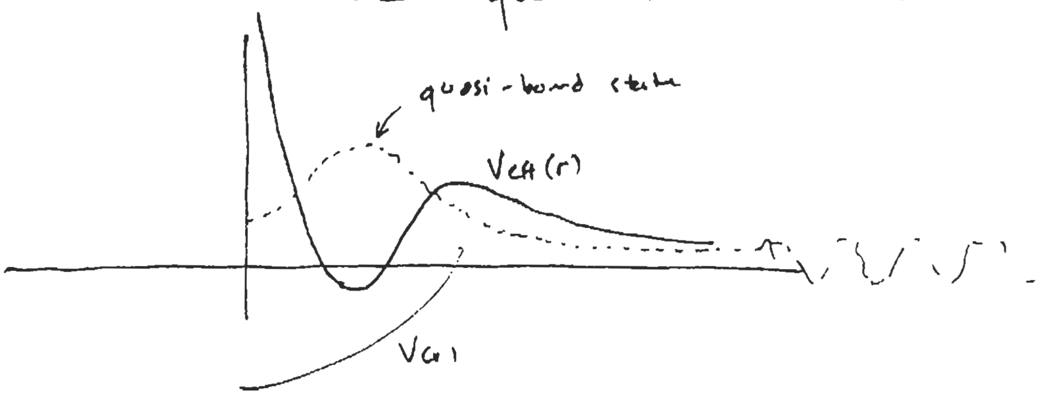
similar story for $e^{2i\delta_\ell}$

Resonances

For any partial wave ℓ, eff. potential is

$$V_{eff}(r) = V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}$$

Potential can have "quasi-bound" state



Bd state for general ℓ (low E)

$$\underset{r>R}{A(r)} \propto h_\ell^+(i\alpha r) \xrightarrow[r\to\infty]{} \frac{e^{-\alpha r}}{r}$$

$$\beta_\ell = \frac{r\frac{d}{dr}h_\ell^+}{h_\ell^+} \Rightarrow -\ell-1 \qquad \left(h_\ell^+(\rho) \sim -\frac{(2\ell-1)!!}{\rho^{\ell+1}}\right)$$

Bound state: $\beta_\ell(i\alpha) = -\ell-1$

Raise potential slightly, becomes quasi-bound

$$\beta_\ell(k) = -\ell-1 \qquad \left(\begin{array}{l} j_\ell \sim \frac{\rho^\ell}{(2\ell+1)!!} \\ n_\ell \sim -\frac{(2\ell-1)!!}{\rho^{\ell+1}} \end{array}\right)$$

Expand

$$\tan\delta_\ell = \frac{\rho j_\ell' - \beta_\ell j_\ell}{\rho n_\ell' - \beta_\ell n_\ell}$$

$$\cong + \frac{\rho^{2\ell+1}}{(2\ell+1)!!\,(2\ell-1)!!}\left[\frac{\ell-\beta_\ell}{\ell+1+\beta_\ell}\right] + \cdots$$

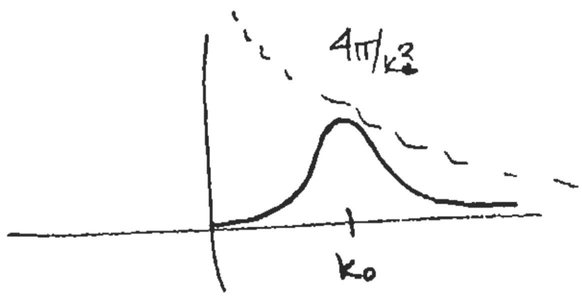
~~[illegible]~~

Shift in δ_ℓ ⟶ time lag in reflection due to resonance

when $\delta_\ell = \pi/2$, $\sin^2\delta_\ell = 1$,

max of σ_ℓ is saturated

$$\sigma_\ell(k_0) = \frac{4\pi}{k_0^2}(2\ell+1)$$



shape of resonance:

$$\tan\delta_\ell \equiv \frac{1}{2}\frac{\Gamma_\ell}{k_0-k}$$

$$\frac{1}{2}\Gamma_\ell = -\frac{(k_0R)^{2\ell+1}}{[(2\ell-1)!!]^2 \left.\frac{d\beta_\ell}{dk}\right|_{k_0}}$$

More generally ——— $\Gamma = -\dfrac{2}{d\cot\delta_\ell/dk}$

$$\Rightarrow \sigma_\ell = \frac{4\pi(2\ell+1)}{k_0^2}\sin^2\delta_\ell$$

[note: in bk def. rel. to E: $(d\cot\delta_\ell/dE)$]

$$= \frac{4\pi(2\ell+1)}{k_0^2}\,\frac{\frac{1}{4}\Gamma_\ell^2}{(k_0-k)^2+\Gamma_\ell^2/4} \quad \text{(Breit-Wigner)}$$

Consider
$$k f_\ell = \frac{1}{\cot\delta - i} = \frac{1}{\frac{k_0 - k}{\Gamma_\ell/2} - i} = \frac{\Gamma_\ell/2}{k_0 - k - i\Gamma_\ell/2}$$

so $k f_\ell$ has pole at $k = k_0 - i\Gamma_\ell/2$

rapid change in σ_ℓ from proximity to the pole.

10.8 Coulomb Scattering

Radial Schrödinger eqn for potential $U(r)$
$$u_\ell'' + \left(k^2 - U(r) - \frac{\ell(\ell+1)}{r^2}\right) u_\ell = 0$$

Asymptotic form $r \to \infty$

Assume $u_\ell = \alpha\, e^{\pm ikr} e^{\int_{r_0}^{r} F(r)\,dr}$

$$\Rightarrow F^2 \pm 2ikF + F' - U(r) - \frac{\ell(\ell+1)}{r^2} = 0$$

If $U(r) = \frac{C}{r}$,
$$F = \pm \frac{C}{2ikr} + \mathcal{O}\left(\frac{1}{r^2}\right)$$