MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.282 | April 16, 2003

Quiz 2

Name SOLUTIONS **(please print)**
Last **First**

1. Work any **7** of the **10** problems - indicate clearly which 7 you want graded.
2. Spend up to 20 minutes reviewing all the problems, and select 7.
3. Immediately after this you have two continuous hours to work the problems.
4. All problems are worth 14 points.
5. Closed book exam; you may use two pages of notes, and a calculator.
6. Wherever possible, try to solve the problems using general analytic expressions. Plug in numbers only as a last step.
7. If you have any questions, e-mail the Instructor.

Time Started: Time Stopped:

Signature ______________________________

Problem	Grade	Grader
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
Total		

USEFUL CONSTANTS

Constant	cgs units		mks units	
c (speed of light)	3×10^{10}	cm/sec	3×10^{8}	m/sec
G (gravitation constant)	7×10^{-8}	dyne-cm^2/g^2	7×10^{-11}	N-m^2/kg^2
k (Boltzmann's constant)	1.4×10^{-16}	erg/K	1.4×10^{-23}	J/K
h (Planck's constant)	6.6×10^{-27}	erg-sec	6.6×10^{-34}	J-sec
$m_{\rm proton}$	1.6×10^{-24}	g	1.6×10^{-27}	kg
eV (electron Volt)	1.6×10^{-12}	erg	1.6×10^{-19}	J
$M_\odot$ (solar mass)	2×10^{33}	g	2×10^{30}	kg
$L_\odot$ (solar luminosity)	4×10^{33}	erg/sec	4×10^{26}	J/sec
$R_\odot$ (solar radius)	7×10^{10}	cm	7×10^{8}	m
σ (Stefan-Boltzmann cons)	6×10^{-5}	erg/cm^2-sec-K^4	6×10^{-8}	J/m^2-sec-K^4
Å(Angstrom)	10^{-8}	cm	10^{-10}	m
km (kilometer)	10^{5}	cm	10^{3}	m
pc (parsec)	3×10^{18}	cm	3×10^{16}	m
kpc (kiloparsec)	3×10^{21}	cm	3×10^{19}	m
Mpc (megaparsec)	3×10^{24}	cm	3×10^{22}	m
year	3×10^{7}	sec	3×10^{7}	sec
day	86400	sec	86400	sec
AU	1.5×10^{13}	cm	1.5×10^{11}	m
$1'$ (arc minute)	$1/3400$	rad	$1/3400$	rad
$1''$ (arc second)	$1/200,000$	rad	$1/200,000$	rad

Problem 1

A very hot star is detected in the galaxy M31 located at a distance of 800 kpc. The star has a temperature $T = 6 \times 10^5$ K and a produces a flux of 10^{-12} ergs cm^{-2} sec^{-1} (10^{-9} W m^{-2}) at the Earth. Treat the star's surface as a blackbody radiator.

a. Find the luminosity of the star.

$$L = 4\pi D^2 \mathrm{Flux}_{\mathrm{Earth}} = 10^{-12} 4\pi (800 \times 3 \times 10^{21})^2 = 7 \times 10^{37} \mathrm{erg} \cdot \mathrm{s}^{-1}$$

There was an error in the statement of the problem in that the mks units do not match the cgs value. In this case, a valid answer would be based on a flux that is 10^6 times larger.

b. Compute the star's radius.

$$R = \left(L/4\pi\sigma T^4\right)^{1/2} = 8.7 \times 10^8 \,\mathrm{cm} = 0.012 \ R_\odot$$

(The radius would be 10^3 times larger if you used the mks flux.)

c. At what wavelength is the peak of the emitted radiation?
Use the Wien displacement law:

$$\lambda_{\mathrm{max}} = 0.29/T \,\mathrm{cm} = 48 \,\text{Å}$$

Problem 2

Spectra of a binary star system are recorded over a time interval of months. It is found that the wavelengths of the lines from one of the stars vary sinusoidally with a period of 7 days. The half amplitude of the wavelength shifts is $\Delta\lambda/\lambda = 0.001$. The spectrum from the other star is not detected. Exactly what information can be learned about the masses of the two stars? Be quantitative. (Hint: your answer should include the concept and numerical value of the "mass function" for this system.)

$$\frac{\Delta\lambda}{\lambda} = \frac{v_1'}{c} = 10^{-3}$$

$$v_1' = 3 \times 10^7\,\mathrm{cm}\cdot\mathrm{s}^{-1}$$

However, the "velocity" v_1' is only the projection of the star's velocity along the line of sight. If i is the orbital inclination angle, then $v_1' = v_1 \sin i$.

$$2\pi a_1 \sin i = (v_1 \sin i)P = (3 \times 10^7\,\mathrm{cm}\cdot\mathrm{s}^{-1})(7\,\mathrm{days})$$

$$a_1 \sin i = 2.9 \times 10^{12}\,\mathrm{cm}$$

From Kepler's Third Law, we can construct the mass function,

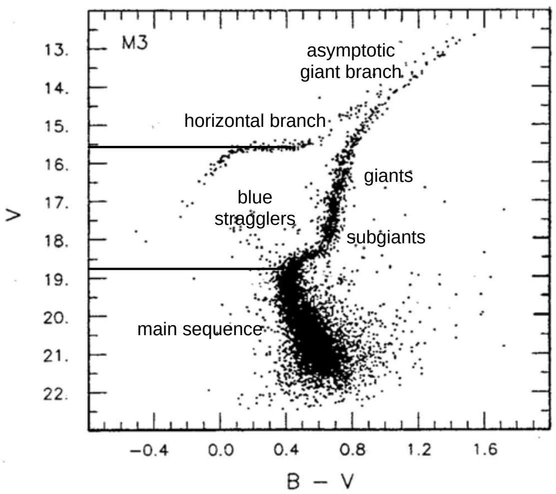
$$f(M) = \frac{M_2 \sin^3 i}{\left(1 + \frac{M_1}{M_2}\right)^2} = \frac{(2\pi)^2}{G}\frac{(a_1 \sin i)^3}{P^2},$$

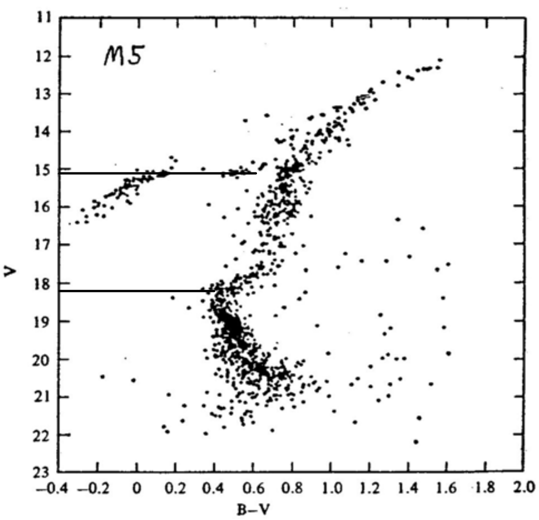
which depends only upon known quantities. Since $\sin^3 i/(1 + M_1/M_2)^2 < 1$, $f(M)$ is a lower limit on M_2.

$$f(M) = 3.9 \times 10^{34}\,\mathrm{g} = 19.5 M_\odot$$

Problem 3

The figures below show H–R diagrams for two globular clusters: M3 and M5. (V is the apparent visual magnitude.)





a. Identify the various stellar evolutionary phases seen on the H–R diagram for M3. You may write your answers directly on the plot.
b. Find the relative distances to the two clusters, i.e., $d_{\mathrm{M3}}/d_{\mathrm{M5}}$.

The horizontal branch lies at magnitude 15.5 for M3 and 15.1 for M5, and the main sequence turnoffs are at 18.8 and 18.25. These give an average magnitude difference of 0.47.

$$0.47 = 5 \log \left(\frac{d_{\mathrm{M3}}}{d_{\mathrm{M5}}} \right)$$

$$\frac{d_{\mathrm{M3}}}{d_{\mathrm{M5}}} = 1.24$$

c. What are the predominant nuclear reactions powering the different evolutionary phases, and where are they taking place within the star (e.g., in a shell or in the core)? You need not write down specific nuclear reactions, simply state what elements are being "burned."

main sequence: H → He in core
giants: H → He in shell
asymptotic giant branch: H → He and He → C in shell
horizontal branch: He → C and O in core

d. How many main-sequence stars with $V = 20$ equal the brightness of a single giant with $V = 12.5$?

$$V = -2.5 \log \left(\frac{F}{F_0} \right)$$

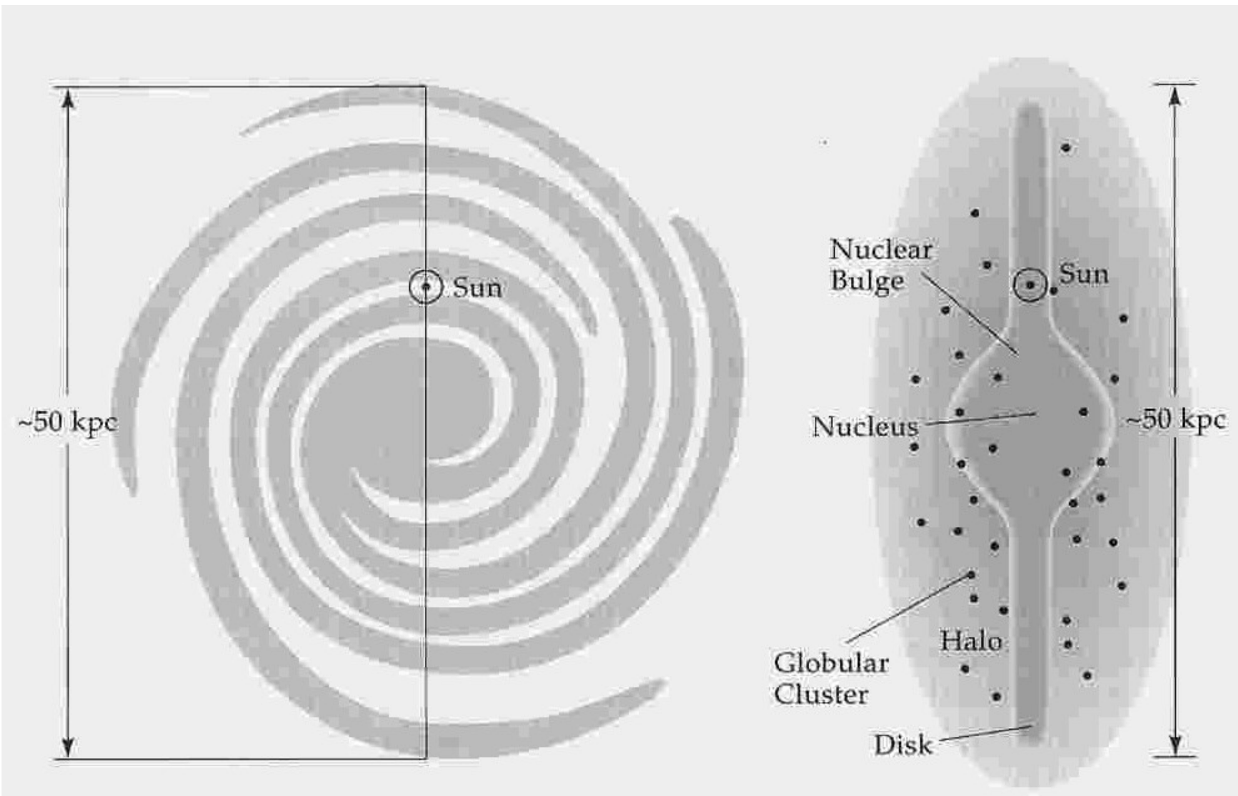
One star has $V = 20$, so $20 = -2.5 \log\left(\frac{F}{F_0}\right)$, or $F/F_0 = 10^{-8}$.
N stars have normalized flux NF/F_0 and $V = 12.5$.

$$V = 12.5 = -2.5 \log\left(\frac{NF}{F_0}\right) = -2.5 \log\left(\frac{F}{F_0}\right) - 2.5 \log N = 20 - 2.5 \log N$$

$$N = 10^3$$

Problem 4

Make a sketch of our Galaxy (top view and side view), including any qualitative structures contained therein and the regions that different types of objects occupy. Mark any size scales you know. If the Sun (more precisely, the local standard of rest) is orbiting the Galactic center at 220 km/sec, estimate the mass of the Galaxy interior to the Sun's orbit (in units of $M_{\odot}$). Choose whatever distance to the Galactic center you happen to know.



An estimate of the mass contained within the Sun's orbit around the Galaxy is:

$$\frac{GM(<R)}{R^2} \simeq \frac{v_{\odot}^2}{R}$$
$$M(<R) \simeq \frac{(220 \times 10^5)^2 (8 \text{ kpc})}{G}$$
$$M(<R) \simeq 1.7 \times 10^{44} \text{ grams} \simeq 9 \times 10^{10}\ M_{\odot}$$

Problem 5

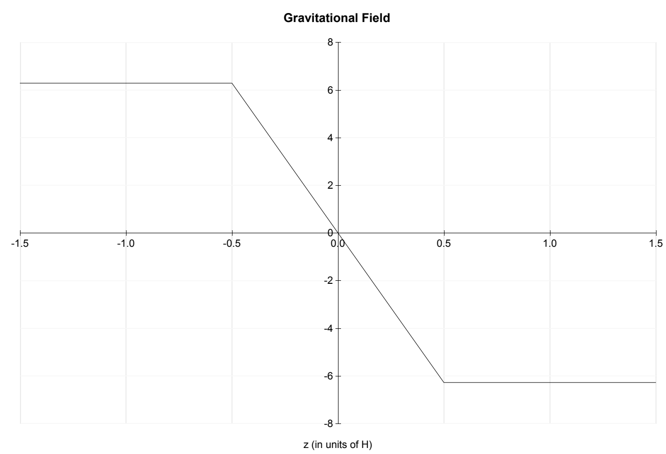
Suppose we model the Galactic disk as a uniform slab of material with *full* thickness H in the $\hat{z}$ direction, and effectively infinite in the $\hat{x}$ and $\hat{y}$ directions. The density of matter for $|z| < H/2$ is ρ_0, while for $|z| > H/2$, ρ_0 drops abruptly to 0. Find and sketch the gravitational field for all values of z (both + and −). (Hint: use Gauss' law, but applied to gravity.)

Imagine drawing a Gaussian cylinder with ends of area A located at $\pm z$ (i.e. symmetric placed on either side of the mid-plane). Gauss' Law for gravity says

$$\int_{\text{cylinder}} \vec{g} \cdot d\vec{A} = -4\pi G \int_{\text{interior}} \rho \, dV.$$

The left-hand-side is always $-2gA$; the right-hand-side is $-4\pi G\rho_0(2zA)$ if $z < H/2$ and $-4\pi G\rho_0(HA)$ if $z > H/2$. So $|\vec{g}| = 4\pi G\rho_0|z|$ inside the disk and $|\vec{g}| = 2\pi G\rho_0 H$ outside.

This plot shows the z-component of $\vec{g}$, in units of $G\rho_0 H$.



Problem 6

Recent measurements find that the Oort A and Oort B coefficients for our Galaxy are 14 km $\sec^{-1}$ kpc^{-1} and -12 km $\sec^{-1}$ kpc^{-1}, respectively. (Recall that $A \equiv -\frac{1}{2}R(d\omega/dR)$ and $B \equiv A - \omega$.)

a. What assumptions go into the Oort equations from which the A and B coefficients are derived?

$\omega(R)$ is a monotonically decreasing function of R
All orbits are circular
Only stars out to a distance of $\sim$ 1 kpc are used

b. What measurements are needed to determine A and B?

Measurements of v_r, v_t (proper motion), and the distance to the stars are needed as a function of l.

c. Use the above values of A and B to find the rotation frequency (or period) of the Sun around the Galaxy.

$$B = A - \omega$$

$$\omega = A - B = 14 - (-12) = 26 \mathrm{kmsec}^{-1}\mathrm{kpc}^{-1}$$

$$\omega = 26 \text{ km/sec}/3 \times 10^{16} \text{ km}$$

$$\omega = 8.7 \times 10{-}16 \text{ radians/sec} = \frac{2\pi}{P}$$

$$P \simeq 7 \times 10^{15} \text{ sec} = 250 \text{ million years}$$

d. What can we learn about $\omega(R)$ for the Galaxy from the Oort method?

$$A = -\frac{1}{2}\left(\frac{R}{\omega}\right)\left(\frac{d\omega}{dR}\right)\omega \simeq 14$$

$$\frac{d\ln R}{d\ln\omega} = -\frac{28}{\omega} = -\frac{28}{26} \simeq -1$$

Problem 7

Short answer questions:

a. Explain why the magnitude difference, $U - B$ of a star is a measure of its temperature.

$$U = -2.5 \log F_U + \text{contsant}$$

$$B = -2.5 \log F_B + \text{sameconstant}$$

$$U - B = -2.5 \log \left(\frac{F_U}{F_B} \right)$$

So $U - B$ measures the ratio of fluxes at two different wavelengths, and thereby the temperature.

b. Explain what the "Oort limit" is in the context of determining the local mass density in our part of the Galaxy. What is the underlying physics involved?

The "Oort limit" supposes that most of the mass lies close to the galactic plane, and so g is constant. In equilibrium, the particles of matter are exponentially distributed, with a scale height that depends upon $\langle v^2 \rangle$ and g in a known way. By measuring the scale height and $\langle v^2 \rangle$, we determine g, and thus the local mass density.

c. What is a $\log N - \log F$ plot? What slope does it have if the objects being observed have a spherically symmetric distribution?

Such a plot shows the number of objects visible (N) versus their luminosity, both plotted on logarithmic scales. Since the flux from an measured at Earth goes as d^{-2}, the maximum distance out to which objects of a given type may be seen goes as the inverse square root of the luminosity. The volume of space in which we can view the objects (which is proportional to N) goes as the cube of d. So the slope of a log-log plot will be $-3/2$.

Problem 8

We will learn in lecture that for a gas supported by degenerate electron pressure, the pressure is given by:

$$P = K\rho^{5/3}$$

where K is a constant and ρ is the mass density. If a star is totally supported by degenerate electron pressure, use a *dimensional analysis* of the equation of hydrostatic equilibrium:

$$\frac{dP}{dr} = -g\rho$$

to determine how the radius of such a star depends on its mass, M.

$$\frac{K\rho^{5/3}}{R} \simeq \left(\frac{GM}{R^2}\right)\left(\frac{M}{R^3}\right)$$

$$\rho \sim \frac{M}{R^3}$$

$$\frac{KM^{5/3}}{RR^5} \simeq \frac{GM^2}{R^5}$$

$$R \simeq \frac{K}{GM^{1/3}}$$

Problem 9

Take the total energy (potential plus thermal) of the Sun to be given by the simple expression:

$$E \simeq -\frac{GM^2}{R}$$

where M and R are the mass and radius, respectively. Suppose that the energy generation in the Sun were suddenly turned off and the Sun began to slowly contract. During this contraction its mass, M, would remain constant and, to a fair approximation, its surface temperature would also remain constant at ~ 5800 K. Assume that the total energy of the Sun is always given by the above expression, even as R gets smaller. Write down a simple (differential) equation relating the power radiated at Sun's surface with the change in its total energy (using the above expression). Integrate this equation to find the time (in years) for the Sun to shrink to 1/2 its present radius.

$$L = 4\pi\sigma R^2 T^4 = dE/dt = \left(\frac{GM^2}{R^2}\right)\frac{dR}{dt}$$

$$\int_R^{0.5R} \frac{dR}{R^4} = -\int_0^t \frac{4\pi\sigma T^4}{GM^2} dt$$

$$-\frac{1}{3(R/2)^3} + \frac{1}{3R^3} = -\left(\frac{4\pi\sigma T^4}{GM^2}\right) t$$

$$t = \frac{GM^2}{12\pi\sigma T^4}\left(\frac{8}{R^3} - \frac{1}{R^3}\right)$$

$$t = \frac{7GM^2}{12\pi\sigma T^4 R^3} = 2.2 \times 10^{15}\text{sec} = 75 \text{ million years}$$

Problem 10

The principal nuclear reaction in the Sun combines 4 hydrogen nuclei ($_1\mathrm{H}^1$) into a $_2\mathrm{He}^4$ nucleus. Later in the Sun's evolution, three $_2\mathrm{He}^4$ nuclei will be fused into a $_6\mathrm{C}^{12}$ nucleus (via the so-called "triple α" reaction). The atomic mass excesses of $_1\mathrm{H}^1$, $_2\mathrm{He}^4$ and $_6\mathrm{C}^{12}$ are 7.29, 2.42, and 0 MeV, respectively. Use this information to find the efficiency of hydrogen burning compared with He burning. You can define 'efficiency' as the net fractional mass lost in the reaction, $\Delta M/M$. (Note: atomic mass excess = $(M - A{\cdot}\mathrm{amu})c^2$, i.e., the actual mass of an atom minus a reference mass times its integer atomic mass, and then converted to MeV energy units.) Finally, it may be helpful to know that the rest mass energy associated with 1 *amu* is $\sim$ 940 MeV.

$$4_1H^1 \rightarrow {}_2He^4 + \Delta E_1$$

$$4 \times 7.29 - 2.42 = \Delta E_1 = 27 \text{ MeV}$$

$$3_2He^4 \rightarrow {}_6C^{12} + \Delta E_2 = 7.26 \text{ MeV}$$

$$\epsilon_H = \frac{27 \text{ MeV}}{4 \times 940 \text{ MeV}} \simeq 0.007$$

$$\epsilon_{\mathrm{He}} = \frac{7.3 \text{ MeV}}{12 \times 940 \text{ MeV}} \simeq 0.0007$$