MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — October 21, 2012
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QUIZ 1 SOLUTIONS
QUIZ DATE: OCTOBER 18, 2012

PROBLEM 1: SOME SHORT EXERCISES *(30 points)*

(a) *(10 points)* Use index notation to derive a formula for $\vec{\nabla} \times (s\vec{A})$, where s is a scalar field $s(\vec{r})$ and $\vec{A}$ is a vector field $\vec{A}(\vec{r})$.

SOLUTION:

$$\begin{aligned} \left[\vec{\nabla} \times \left(s\vec{A}\right)\right]_i &= \varepsilon_{ijk}\partial_j\left(s\vec{A}\right)_k \\ &= \varepsilon_{ijk}s\partial_j A_k + \varepsilon_{ijk}A_k\partial_j s \\ &= s\vec{\nabla} \times \vec{A} + \vec{\nabla}s \times \vec{A} \, . \end{aligned}$$

(b) *(10 points)* Which of the following vector fields could describe an electric field? Say yes or no for each, and give a very brief reason.

(i) $\vec{E}(\vec{r}) = x\,\hat{e}_x - y\,\hat{e}_y$.

(ii) $\vec{E}(\vec{r}) = y\,\hat{e}_x + x\,\hat{e}_y$.

(iii) $\vec{E}(\vec{r}) = y\,\hat{e}_x - x\,\hat{e}_y$.

SOLUTION: The curl of an electrostatic field must be zero, but otherwise there is no restriction. So the answer follows as

(i) $\vec{\nabla} \times \vec{E}\,(\vec{r}) = \left(\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right)\hat{e}_z + \ldots = \vec{0}$. YES, it describes an electric field.

(ii) $\vec{\nabla} \times \vec{E}\,(\vec{r}) = (1-1)\hat{e}_z = 0$. YES, it describes an electric field.

(iii) $\vec{\nabla} \times \vec{E}\,(\vec{r}) = (-1-1)\hat{e}_z = -2\hat{e}_z$. NO, it does not describe an electric field.

(c) *(10 points)* Suppose that the entire x-z and y-z planes are conducting. Calculate the force $\vec{F}$ on a particle of charge q located at $x = x_0$, $y = y_0$, $z = 0$.

SOLUTION: we need 3 image charges placed as:

$$\begin{aligned} q_1 &= -\,q \quad \text{at} \quad (-x_0, y_0, 0) \\ q_2 &= -\,q \quad \text{at} \quad (x_0, -y_0, 0) \\ q_3 &= +\,q \quad \text{at} \quad (-x_0, -y_0, 0) \, . \end{aligned}$$

Note that q_1 and the original charge give zero potential on the $x = 0$ plane, but allow the potential to vary with x in the $y = 0$ plane. The second image charge, combined with the original charge but ignoring the first image charge, produces a potential that is zero on the $y = 0$ plane, but the potential varies with y on the $x = 0$ plane. The final image charge fixes these remaining problems. For any point on the y-z plane (the $x = 0$ plane) the original charge and q_1 pair to give zero potential, and similarly q_2 and q_3 pair to give zero potential. For points on the x-z plane (where $y = 0$), the original charge and q_2 give canceling potentials, as do q_1 and q_3.

Having found the image charges, we can write the force as

$$\vec{F} = \vec{F}_{q_1} + \vec{F}_{q_2} + \vec{F}_{q_3}$$

where $\vec{F}_{q_i}$ is defined as the force of charge q_i on charge q. The force exerted on the charge q is found to be as:

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{-q^2}{4x_0^2}\hat{x} + \frac{1}{4\pi\epsilon_0}\frac{-q^2}{4y_0^2}\hat{y} + \frac{1}{4\pi\epsilon_0}\frac{q^2}{4(x_0^2+y_0^2)}\frac{x_0\hat{x}+y_0\hat{y}}{\sqrt{x_0^2+y_0^2}}\ .$$

Surprisingly, this question was the one that gave the class the most trouble, with a class average of only 51%. The problem was even illustrated in Lecture Notes 5, on the fourth page of those notes (labeled p. 61). The moral:

PLEASE REVIEW IMAGE CHARGES!

PROBLEM 2: ELECTRIC FIELDS IN A CYLINDRICAL GEOMETRY *(20 points)*

A very long cylindrical object consists of an inner cylinder of radius a, which has a uniform charge density ρ, and a concentric thin cylinder, of radius b, which has an equal but opposite total charge, uniformly distributed on the surface.

(a) *(7 points)* Calculate the electric field everywhere.

(b) *(6 points)* Calculate the electric potential everywhere, taking $V = 0$ on the outer cylinder.

(c) *(7 points)* Calculate the electrostatic energy per unit length of the object.

PROBLEM 2 SOLUTION:

(a) This problem has enough symmetry to allow a solution by Gauss's law. In particular, symmetry considerations imply that the electric field will point radially outward, and will have a magnitude that depends only on the distance from the axis. Following Griffiths, we use s for the distance from the z-axis, and $\hat{s}$ for a unit vector pointing radially outward from the axis, and of course we choose the z-axis to be the axis of the cylindrical object. Then

$$\vec{E} = E(s)\,\hat{s}\ . \tag{2.1}$$

To evaluate $E(s)$, we apply Gauss's law to a Gaussian cylinder of length ℓ, concentric with the z-axis. Then

$$\oint \vec{E}\cdot \mathrm{d}\vec{a} = \frac{Q_{\rm enc}}{\epsilon_0} = 2\pi s\ell\, E(s)\ . \tag{2.2}$$

For $s < a$, the Gaussian cylinder is filled with charge density ρ, so

$$Q_{\rm enc} = \pi s^2 \ell\rho \quad \Longrightarrow \quad E(s) = \frac{\rho s}{2\epsilon_0}. \tag{2.3}$$

For $a < s < b$ the enclosed charge is

$$Q_{\rm enc} = \pi a^2 \ell\rho \quad \Longrightarrow \quad E(s) = \frac{\rho a^2}{2\epsilon_0 s}\ . \tag{2.4}$$

Finally, for $s > b$ the enclosed charge is zero, so $E(s) = 0$. Putting this together,

$$\boxed{\vec{E} = \frac{\rho}{2\epsilon_0}\begin{cases} s\,\hat{s} & \text{if } s < a \\ \dfrac{a^2}{s}\hat{s} & \text{if } a < s < b \\ \vec{0} & \text{if } s > a\ \ . \end{cases}} \tag{2.5}$$

(b) To find the potential from the electric field, we can use

$$V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{r}\,')\cdot \mathrm{d}\vec{\ell}' \tag{2.6}$$

from the formula sheet. Since the line integrals that define the electric potential are path-independent, we can choose to integrate only over radial paths. For $s > b$ we clearly have $V(s) = 0$, since the absence of an electric field in this region implies that $V = \text{const}$, and $V = 0$ at $s = b$. Then, for $a \le s \le b$,

$$V(s) = V(s{=}b) - \int_b^s \vec{E}\cdot \mathrm{d}\vec{\ell} = 0 + \int_s^b \vec{E}\cdot \mathrm{d}\vec{\ell} = \frac{\rho}{2\epsilon_0}\int_s^b \frac{a^2}{s}\,\mathrm{d}s = \frac{\rho a^2}{2\epsilon_0}\ln\frac{b}{s}\ . \tag{2.7}$$

This is valid down to $s = a$, so $V(a) = \frac{\rho a^2}{2\epsilon_0 \ln(b/a)}$, and then for $s \le a$,

$$\begin{aligned} V(s) &= V(s{=}a) - \int_a^s \vec{E}\cdot d\vec{\ell} = \frac{\rho a^2}{2\epsilon_0 \ln(b/a)} + \frac{\rho}{2\epsilon_0}\int_s^a s\,ds \\ &= \frac{\rho a^2}{2\epsilon_0}\ln(b/a) + \frac{\rho}{4\epsilon_0}(a^2 - s^2) = \frac{\rho}{4\epsilon_0}\left[2a^2\ln(b/a) + a^2 - s^2\right] \; . \end{aligned} \tag{2.8}$$

Putting these together

$$\boxed{V(s) = \frac{\rho}{4\epsilon_0}\begin{cases} 0 & \text{if } s > b \\ 2a^2 \ln\frac{b}{s} & \text{if } a < s < b \\ 2a^2\ln(b/a) + a^2 - s^2 & \text{if } s < a \; . \end{cases}} \tag{2.9}$$

(c) To find the electrostatic energy, we can use either

$$W = \frac{1}{2}\epsilon_0 \int \left|\vec{E}\right|^2 d^3x \tag{2.10}$$

or

$$W = \frac{1}{2}\int \rho(\vec{r})V(\vec{r})\, d^3x \; . \tag{2.11}$$

Using Eq. (2.10) with (2.5),

$$W = \frac{1}{2}\epsilon_0\left\{\int_0^a \left(\frac{\rho}{2\epsilon_0}\right)^2 s^2\, 2\pi s\, ds + \int_a^b \left(\frac{\rho}{2\epsilon_0}\right)^2 \frac{a^4}{s^2}\, 2\pi s\, ds\right\}\ell \; , \tag{2.12}$$

so

$$\begin{aligned} \frac{W}{\ell} &= \frac{\pi\rho^2}{4\epsilon_0}\left\{\int_0^a s^3\, ds + \int_a^b \frac{a^4}{s}\, ds\right\} \\ &= \frac{\pi\rho^2}{4\epsilon_0}\left\{\frac{1}{4}a^4 + a^4\ln\left(\frac{b}{a}\right)\right\} \\ &= \boxed{\frac{\pi\rho^2 a^4}{16\epsilon_0}\left[1 + 4\ln\left(\frac{b}{a}\right)\right] \; .} \end{aligned} \tag{2.13}$$

By using Eqs. (2.11) with (2.9), we first note that $V = 0$ on the outer cylinder, so we get a contribution only by integrating over the inner cylinder:

$$\begin{aligned}
\frac{W}{\ell} &= \frac{\rho^2}{8\epsilon_0} \int_0^a \left[2a^2 \ln\left(\frac{b}{a}\right) + a^2 - s^2 \right] 2\pi s \, \mathrm{d}s \\
&= \frac{\pi\rho^2}{4\epsilon_0} \left\{ \left[2a^2 \ln\left(\frac{b}{a}\right) + a^2 \right] \int_0^a s \, \mathrm{d}s - \int_0^a s^3 \, \mathrm{d}s \right\} \\
&= \frac{\pi\rho^2}{4\epsilon_0} \left\{ \left[2a^2 \ln\left(\frac{b}{a}\right) + a^2 \right] \left(\frac{a^2}{2}\right) - \left(\frac{a^4}{4}\right) \right\} \qquad (2.14) \\
&= \boxed{ \frac{\pi\rho^2 a^4}{16\epsilon_0} \left[1 + 4\ln\left(\frac{b}{a}\right) \right] . }
\end{aligned}$$

PROBLEM 3: MULTIPOLE EXPANSION FOR A CHARGED WIRE *(20 points)*

A short piece of wire is placed along the z-axis, centered at the origin. The wire carries a total charge Q, and the linear charge density λ is an even function of z: $\lambda(z) = \lambda(-z)$. The rms length of the charge distribution in the wire is l_0; i.e.,

$$l_0^2 = \frac{1}{Q} \int_{wire} z^2 \lambda(z) \, \mathrm{d}z \; .$$

(a) *(10 points)* Find the dipole and quadrupole moments for this charge distribution. Note that the dipole and quadrupole moments are defined on the formula sheets as

$$p_i = \int d^3x \, \rho(\vec{r}) \, x_i \; ,$$

$$Q_{ij} = \int d^3x \, \rho(\vec{r}) (3x_i x_j - \delta_{ij} |\vec{r}|^2) \; .$$

(b) *(10 points)* Give an expression for the potential $V(r, \theta)$ for large r, including all terms through the quadrupole contribution.

PROBLEM 3 SOLUTION:

(a) *(10 points)* The dipole moment is defined as

$$p_i = \int \mathrm{d}^3x\, \rho(\vec{r})\, x_i \ .$$

In this case the x and y components are zero, since $x_1 = x_2 = 0$ for the wire which runs along z-axis. The z component of the dipole moment is p_z, given by

$$p_z = \int_{wire} \lambda(z)\, z\, \mathrm{d}z \ ,$$

where $\mathrm{d}^3x\, \rho(\vec{r})$ from the general formalism was replaced by $\lambda(z)\,\mathrm{d}z$. This integration also yields zero since $\lambda(z)$ being an even function makes $\lambda(z)\,z$ an odd function. Therefore the integral gives zero. The dipole moment is found to be

$$\boxed{\vec{p} = 0 \ .}$$

The quadrupole moments are defined as,

$$Q_{ij} = \int \mathrm{d}^3x \rho(\vec{r})(3x_i x_j - \delta_{ij}|\vec{r}|^2) \ .$$

Since the wire runs along z-axis we again have $x_1 = 0$ and $x_2 = 0$, and we also have $|\vec{r}| = |z|$ on the wire. Using the rms length of the charge distribution, $\int_{wire} z^2 \lambda(z) dz = Ql_0^2$, we find the quadrupole moment as

$$\boxed{\begin{aligned} Q_{xx} = Q_{yy} &= \int_{wire} dz \lambda(z)(-z^2) = -Ql_0^2 \ , \\ Q_{zz} &= \int_{wire} dz \lambda(z)(3z^2 - z^2) = 2Ql_0^2 \ , \\ Q_{xy} &= Q_{yx} = Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0 \ . \end{aligned}}$$

(b) *(10 points)* We use the formula for the multipole expansion of the potential on formula sheet,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\Big[\frac{Q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{1}{2}\frac{\hat{r}_i\hat{r}_j}{r^3}Q_{ij} + \ldots\Big] \ ,$$

where Q, p_i and Q_{ij} are given in part (a). The $\hat{r}$ direction is

$$\hat{r} = \sin\theta\cos\phi\hat{e}_x + \sin\theta\sin\phi\hat{e}_y + \cos\theta\hat{e}_z = r_x\hat{x} + r_y\hat{y} + r_z\hat{z}$$

Then performing the sum, we find the potential $V(r,\theta)$ as

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{1}{2}\frac{r_x r_x}{r^3}Q_{xx} + \frac{1}{2}\frac{r_y r_y}{r^3}Q_{yy} + \frac{1}{2}\frac{r_z r_z}{r^3}Q_{zz}\dots\right]$$

Up to the quadrupole term,

$$\begin{aligned} V(r,\theta) &= \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{Ql_0^2}{2r^3}(-\sin^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi + 2\cos^2\theta)\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{Ql_0^2}{2r^3}(-\sin^2\theta + 2\cos^2\theta)\right] \\ &= \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{Ql_0^2}{2r^3}(3\cos^2\theta - 1)\right]\ .} \end{aligned}$$

PROBLEM 4: A SPHERICAL SHELL OF CHARGE *(30 points)*

(a) *(10 points)* A spherical shell of radius R, with an unspecified surface charge density, is centered at the origin of our coordinate system. The electric potential on the shell is known to be

$$V(\theta,\phi) = V_0 \sin\theta\cos\phi\ ,$$

where V_0 is a constant, and we use the usual polar coordinates, related to the Cartesian coordinates by

$$\begin{aligned} x &= r\sin\theta\cos\phi\ , \\ y &= r\sin\theta\sin\phi\ , \\ z &= r\cos\theta\ . \end{aligned}$$

Find $V(r,\theta,\phi)$ everywhere, both inside and outside the sphere. Assume that the zero of V is fixed by requiring V to approach zero at spatial infinity. (*Hint:* this problem can be solved using traceless symmetric tensors, or if you prefer you can use standard spherical harmonics. A table of the low-ℓ Legendre polynomials and spherical harmonics is included with the formula sheets.)

(b) *(10 points)* Suppose instead that the potential on the shell is given by

$$V(\theta,\phi) = V_0 \sin^2\theta\sin^2\phi\ .$$

Again, find $V(r,\theta,\phi)$ everywhere, both inside and outside the sphere.

(c) *(10 points)* Suppose instead of specifying the potential, suppose the surface charge density is known to be

$$\sigma(\theta,\phi) = \sigma_0 \sin^2\theta\sin^2\phi\ .$$

Once again, find $V(r,\theta,\phi)$ everywhere.

PROBLEM 4 SOLUTION:

This problem can be solving using either traceless symmetric tensors or the more standard spherical harmonics. I will show the solution both ways, starting with the simplier derivation in terms of traceless symmetric tensors.

(a) *(10 points)* We exploit the fact that the most general solution to Laplace's equation can be written as a sum of terms of the form

$$\left(r^\ell \text{ or } \frac{1}{r^{\ell+1}} \right) C^{(\ell)}_{i_1 \ldots i_\ell} \, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell} \ , \tag{4.1}$$

where $C^{(\ell)}_{i_1 \ldots i_\ell}$ is a traceless symmetric tensor. In this case we only need an $\ell = 1$ term, since

$$F_a(\theta, \phi) \equiv \sin\theta \cos\phi = \frac{x}{r} = \hat{x}_i \hat{n}_i \ . \tag{4.2}$$

For $\ell = 1$ the radial function must be r or $1/r^2$. For $r < R$ the $1/r^2$ option is excluded, since it is infinite at $r = 0$, so the solution is

$$\begin{aligned} V(\vec{r}) &= V_0 \frac{r}{R} F_a(\theta, \phi) \\ &= \boxed{V_0 \frac{r}{R} \hat{x}_i \hat{n}_i} \quad \text{or} \quad \boxed{V_0 \frac{r}{R} \sin\theta \cos\phi \ .} \end{aligned} \tag{4.3}$$

Note that the factor $(1/R)$ was chosen to match the boundary condition at $r = R$. For $r > R$ the term proportional to r is excluded, because it does not approach zero as $r \to \infty$, so only the $1/r^2$ option remains, and the solution is

$$\begin{aligned} V(\vec{r}) &= V_0 \left(\frac{R}{r} \right)^2 F_a(\theta, \phi) \\ &= \boxed{V_0 \left(\frac{R}{r} \right)^2 \hat{x}_i \hat{n}_i} \quad \text{or} \quad \boxed{V_0 \left(\frac{R}{r} \right)^2 \sin\theta \cos\phi \ .} \end{aligned} \tag{4.4}$$

(b) *(10 points)* This is in principle the same problem as in part (a), with a slightly more complicated angular pattern. In this case

$$F_b(\theta, \phi) \equiv \sin^2\theta \sin^2\phi = \frac{y^2}{r^2} = (\hat{y} \cdot \hat{n})^2 = \hat{y}_i \hat{y}_j \, \hat{n}_i \hat{n}_j \ . \tag{4.5}$$

This is not quite the expansion in traceless symmetric tensors that we want, because $\hat{y}_i\hat{y}_j$ is not traceless, but instead has trace $\delta_{ij}\hat{y}_i\hat{y}_j = \hat{y}\cdot\hat{y} = 1$. However, we can easily make it traceless by subtracting $\frac{1}{3}\delta_{ij}$, writing

$$F_b(\theta,\phi) = \left[\hat{y}_i\hat{y}_j - \frac{1}{3}\delta_{ij}\right]\hat{n}_i\hat{n}_j + \frac{1}{3} \ . \tag{4.6}$$

To simplify the notation of what follows, I define

$$F_2(\theta,\phi) = \left[\hat{y}_i\hat{y}_j - \frac{1}{3}\delta_{ij}\right]\hat{n}_i\hat{n}_j = \sin^2\theta\sin^2\phi - \frac{1}{3} \ , \tag{4.7}$$

and

$$F_0(\theta,\phi) = \frac{1}{3} \ , \tag{4.8}$$

so

$$F_b(\theta,\phi) = F_2(\theta,\phi) + F_0(\theta,\phi) \ , \tag{4.9}$$

where F_2 and F_0 refer to the $\ell = 2$ and $\ell = 0$ parts. To construct the potential, the $\ell = 2$ term can be multiplied by r^2 or $1/r^3$, where the second is excluded for $r < R$ and the first is excluded for $r > R$. The $\ell = 0$ term can be multiplied by 1 or $1/r$, where the second is excluded for $r < R$ and the first is excluded for $r > R$. Thus, for $r < R$ we have

$$V(\vec{r}) = V_0\left[\left(\frac{r}{R}\right)^2 F_2(\theta,\phi) + F_0(\theta,\phi)\right] \tag{4.10a}$$

$$= \boxed{V_0\left[\left(\frac{r}{R}\right)^2 \left(\hat{y}_i\hat{y}_j - \tfrac{1}{3}\delta_{ij}\right)\hat{n}_i\hat{n}_j + \tfrac{1}{3}\right]} \quad \text{or} \tag{4.10b}$$

$$\boxed{V_0\left[\left(\frac{r}{R}\right)^2 \left(\sin^2\theta\sin^2\phi - \tfrac{1}{3}\right) + \tfrac{1}{3}\right] \ .} \tag{4.10c}$$

For $r > R$ we have

$$V(\vec{r}) = V_0\left[\left(\frac{R}{r}\right)^3 F_2(\theta,\phi) + \left(\frac{R}{r}\right) F_0(\theta,\phi)\right] \tag{4.11a}$$

$$= \boxed{V_0\left[\left(\frac{R}{r}\right)^3 \left(\hat{y}_i\hat{y}_j - \tfrac{1}{3}\delta_{ij}\right)\hat{n}_i\hat{n}_j + \tfrac{1}{3}\left(\frac{R}{r}\right)\right]} \quad \text{or} \tag{4.11b}$$

$$\boxed{V_0\left[\left(\frac{R}{r}\right)^3 \left(\sin^2\theta\sin^2\phi - \tfrac{1}{3}\right) + \tfrac{1}{3}\left(\frac{R}{r}\right)\right] \ .} \tag{4.11c}$$

(c) In this case we are given $\sigma(\theta, \phi)$ instead of the potential at $r = R$, so we need to make use of the fact that the surface charge density is related to the discontinuity in the radial component of the electric field. From Gauss's law, we know that

$$E_r(r{=}R{+}) - E_r(r{=}R{-}) = \frac{\sigma}{\epsilon_0} \ . \tag{4.12}$$

From the previous part, we know that we can write the potential as

$$V(\vec{r}) = \begin{cases} A\left(\frac{r}{R}\right)^2 F_2(\theta,\phi) + BF_0(\theta,\phi) & \text{for } r < R \\ A'\left(\frac{R}{r}\right)^3 F_2(\theta,\phi) + B'\left(\frac{R}{r}\right) F_0(\theta,\phi) & \text{for } r > R \ , \end{cases} \tag{4.13}$$

where A, B, A', and B' are as yet unknown constants. For the potential to be continuous at $r = R$ (potentials are always continuous if the electric fields are finite), we require $A' = A$ and $B' = B$; the terms must match individually, since F_0 and F_2 are orthogonal to each other.

The surface charge density can be written as

$$\sigma(\theta,\phi) = \sigma_0 F_b(\theta,\phi) = \sigma_0\big(F_2(\theta,\phi) + F_0(\theta,\phi)\big) \ , \tag{4.14}$$

so we can write the discontinuity equation (4.12) as

$$\begin{aligned} &-\frac{\partial V}{\partial r}(r{=}R{+}) + \frac{\partial V}{\partial r}(r{=}R{-}) = \\ &\quad \frac{3A}{R}F_2(\theta,\phi) + \frac{B}{R}F_0(\theta,\phi) + \frac{2A}{R}F_2(\theta,\phi) = \frac{\sigma_0}{\epsilon_0}\left(F_2(\theta,\phi) + F_0(\theta,\phi)\right) \ . \end{aligned} \tag{4.15}$$

Again, since F_0 and F_2 are orthogonal, the coefficients must match for each of them, leading to

$$A = \frac{R\sigma_0}{5\epsilon_0} \ , \qquad B = \frac{R\sigma_0}{\epsilon_0} \ . \tag{4.16}$$

Inserting these coefficients into Eq. (4.13), we find

$$\boxed{V(\vec{r}) = \frac{R\sigma_0}{5\epsilon_0}\begin{cases} \left(\frac{r}{R}\right)^2\left(\sin^2\theta\sin^2\phi - \frac{1}{3}\right) + \frac{5}{3} & \text{for } r < R \\ \left(\frac{R}{r}\right)^3\left(\sin^2\theta\sin^2\phi - \frac{1}{3}\right) + \frac{5}{3}\left(\frac{R}{r}\right) & \text{for } r > R \ . \end{cases}} \tag{4.17}$$

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For those who enjoy looking up functions in tables and manipulating complicated expressions involving factors or $\sqrt{4\pi}$, the method of spherical harmonics is the ideal choice. Most students in the class chose this option.

(a) This part is pretty straightforward, whether one uses traceless symmetric tensors or spherical harmonics. Using the table in the formula sheets, and the relation

$$Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi) \tag{4.18}$$

from the formula sheet, one can see immediately that

$$F_a(\theta,\phi) \equiv \sin\theta\cos\phi = \sin\theta\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right) = -\sqrt{\frac{2\pi}{3}}\left[Y_{11} - Y_{1,-1}\right] \ . \tag{4.19}$$

The logic is the same as above, and the answer can be written as Eqs. (4.3) and (4.4), or as

$$\boxed{V(\vec{r}) = -V_0\left(\frac{r}{R}\right)\sqrt{\frac{2\pi}{3}}\left[Y_{11}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\right] \qquad \text{for } r < R} \tag{4.20}$$

and

$$\boxed{V(\vec{r}) = -V_0\left(\frac{R}{r}\right)^2\sqrt{\frac{2\pi}{3}}\left[Y_{11}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\right] \qquad \text{for } r > R \ .} \tag{4.21}$$

(b) This time more work is required to express the angular function in terms of spherical harmonics:

$$\begin{aligned}
F_b(\theta,\phi) &= \sin^2\theta\sin^2\phi = \sin^2\theta\left[\frac{e^{i\phi}-e^{-i\phi}}{2i}\right]^2 \\
&= \frac{1}{4}\sin^2\theta\left[2 - e^{2i\phi} - e^{-2i\phi}\right] \\
&= -\sqrt{\frac{2\pi}{15}}\left[Y_{22} + Y_{2,-2}\right] + \frac{1}{2}\sin^2\theta \\
&= -\sqrt{\frac{2\pi}{15}}\left[Y_{22} + Y_{2,-2}\right] + \frac{1}{2}(1-\cos^2\theta) \\
&= -\sqrt{\frac{2\pi}{15}}\left[Y_{22} + Y_{2,-2}\right] - \frac{1}{2}\left(\cos^2\theta - \frac{1}{3}\right) + \frac{1}{3} \\
&= -\sqrt{\frac{2\pi}{15}}\left[Y_{22} + Y_{2,-2}\right] - \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_{20} + \frac{\sqrt{4\pi}}{3}Y_{00} \ .
\end{aligned} \tag{4.22}$$

As before one can separate the $\ell = 0$ and $\ell = 2$ components, writing $F_b(\theta, \phi) = F_2(\theta, \phi) + F_0(\theta, \phi)$, where

$$F_2 = -\sqrt{\frac{2\pi}{15}}\left[Y_{22} + Y_{2,-2}\right] - \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_{20} \tag{4.23}$$

and

$$F_0 = \frac{\sqrt{4\pi}}{3}Y_{00} \ . \tag{4.24}$$

The calculation is then the same as before, so Eqs. (4.10a) and (4.11a) hold for these new expressions for F_2 and F_0. We then conclude that

$$\boxed{V(\vec{r}) = V_0\left\{-\left(\frac{r}{R}\right)^2\left[\sqrt{\frac{2\pi}{15}}\Big(Y_{22} + Y_{2,-2}\Big) + \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_{20}\right] + \frac{\sqrt{4\pi}}{3}Y_{00}\right\}} \tag{4.25}$$

for $r < R$, and

$$\boxed{V(\vec{r}) = V_0\left\{-\left(\frac{R}{r}\right)^3\left[\sqrt{\frac{2\pi}{15}}\Big(Y_{22} + Y_{2,-2}\Big) + \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_{20}\right] + \frac{\sqrt{4\pi}}{3}\left(\frac{R}{r}\right)Y_{00}\right\}} \tag{4.26}$$

for $r > R$. Of course the answers in Eqs. (4.10c) and (4.11c) are still correct, and can be found by replacing the $Y_{\ell m}$'s by their explicit forms.

(c) The calculation is the same as above, except that this time we use Eqs. (4.23) and (4.24) for F_2 and F_0. The result is

$$\boxed{\begin{aligned} V(\vec{r}) = \frac{R\sigma_0}{5\epsilon_0}\Bigg[-\left(\frac{r}{R}\right)^2\left(\sqrt{\frac{2\pi}{15}}\left[Y_{22} + Y_{2,-2}\right] + \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_{20}\right) \\ + \frac{5\sqrt{4\pi}}{3}Y_{00}\Bigg] \end{aligned}} \tag{4.27}$$

for $r < R$, and

$$\boxed{\begin{aligned} V(\vec{r}) = \frac{R\sigma_0}{5\epsilon_0}\Bigg[-\left(\frac{R}{r}\right)^3\left(\sqrt{\frac{2\pi}{15}}\left[Y_{22} + Y_{2,-2}\right] + \frac{1}{3}\sqrt{\frac{4\pi}{5}}Y_{20}\right) \\ + \frac{5\sqrt{4\pi}}{3}\left(\frac{R}{r}\right)Y_{00}\Bigg] \end{aligned}} \tag{4.28}$$

for $r > R$. Eq. (4.17) is still a valid answer, and is what one would find by replacing the $Y_{\ell m}$'s by their explicit values.