# Quantum Physics III (8.06) Spring 2005
# Assignment 3

Feb 15, 2005 **Due WEDNESDAY Feb 23, 2005, 6pm**

### Readings

The current reading assignment is:

- Griffiths Section 10.2.4 is an excellent treatment of the Aharonov-Bohm effect, but ignore the connection to Berry's phase for now. We will come back to this later.
- Cohen-Tannoudji Ch. VI Complement E
- Those of you reading Sakurai should read pp. 130-139.

### Problem Set 3

1. **Gauge Invariance and the Schrödinger Equation (15 points)**

   Recall that if

$$\begin{aligned} \vec{A}'(\vec{x},t) &= \vec{A}(\vec{x},t) - \vec{\nabla} f(\vec{x},t) \\ \phi'(\vec{x},t) &= \phi(\vec{x},t) + \frac{1}{c}\frac{\partial f}{\partial t}(\vec{x},t)\ , \end{aligned} \tag{1}$$

   then $(\vec{A}', \phi')$ and $(\vec{A}, \phi)$ describe the same $\vec{E}$ and $\vec{B}$.

   (a) Write the Schrödinger equation in the "unprimed gauge". Write the Schrödinger equation in the "primed gauge" in terms of unprimed quantities and $f$.

   (b) Show that if $\psi(\vec{x},t)$ solves the Schrödinger equation in the "unprimed gauge", then

$$\psi'(\vec{x},t) \equiv \exp\left(-\frac{iq}{\hbar c} f(\vec{x},t)\right)\psi(\vec{x},t) \tag{2}$$

   solves the Schrödinger equation in the "primed gauge".
   [This means that in quantum mechanics, making a gauge transformation from primed to unprimed gauge means replacing $\vec{A}$, $\phi$ and $\psi$ by $\vec{A}'$, $\phi'$ and $\psi'$, respectively.]

(c) Show that $\langle\psi|\psi\rangle$ and $\langle\psi|x|\psi\rangle$ are the same in the primed and unprimed gauges. This means that the identity operator and the operator $x$ are "gauge invariant operators".

(d) Show that the operator $p$ is *not* gauge invariant, whereas the operator $p - qA/c$ *is* gauge invariant. [Conclusion: the "canonical momentum" $p$ is not a gauge invariant operator, but the "kinetic momentum" — recall that $p - qA/c = mv$ — is a gauge invariant operator.]

(e) Show that the Hamiltonian is a gauge invariant operator.

(f) Suppose that $\psi_n(\vec{x})$ is an eigenstate of the Hamiltonian in the unprimed gauge, with eigenvalue $E_n$. Assume that the gauge transformation function $f$ is time-independent. Show that $\psi'_n(\vec{x})$ is an eigenstate of the Hamiltonian in the primed gauge, with the *same* eigenvalue $E_n$.

Note: you have just shown that the spectrum of energy levels, and the degeneracy of each level, are the same in all gauges.

In 8.05, we said that "physical observables are matrix elements of hermitian operators." We should have said: "physical observables are matrix elements of *gauge invariant* hermitian operators."

2. **Electromagnetic Current Density in Quantum Mechanics (10 points)**

The probability flux in the Schrödinger equation can be identified as the electromagnetic current density, provided the proper attention is paid to the effects of the vector potential. This current density will play a role in our discussion of the quantum Hall effect.

Way back in the 8.04 you derived the probability flux in quantum mechanics:

$$\vec{S}(\vec{x},t) = \frac{\hbar}{m}\mathrm{Im}\left[\psi^*\vec{\nabla}\psi\right] \ .$$

In the presence of electric and magnetic fields, the probability current is modified to

$$\vec{S}(\vec{x},t) = \frac{\hbar}{m}\mathrm{Im}\left[\psi^*\vec{\nabla}\psi\right] - \frac{q}{mc}\psi^*\psi\vec{A} \tag{3}$$

This probability flux is conserved and when multiplied by $q$, the particle's charge, it can be interpreted as the electromagnetic current density, $\vec{j} \equiv q\vec{S}$.

(a) Consider a system defined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}(\vec{x},t)\right)^2 + q\phi(\vec{x},t). \tag{4}$$

The corresponding time dependent Schrödinger equation in the presence of (possibly time dependent) electric and magnetic fields is:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}(\vec{x},t)\right)^2 \psi(\vec{x},t) + q\phi(\vec{x},t)\psi(\vec{x},t)\ . \qquad (5)$$

Derive the expression eq. (3) for the probability flux, using the following steps:

- Choose to work in a gauge where $\vec{\nabla}\cdot\vec{A} = 0$.[1]
- Write down the complex conjugate of eq. (5), multiply by $\psi$, and subtract the two equations.
- The resulting equation can be written in the form:

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{S}$$

  Show that $\rho = \psi^*\psi$ and that $\vec{S}$ is given by eq. (3)

(b) Assuming that $\psi$ has units $1/l^{3/2}$ as one would expect from the normalization condition, $\int d^3x\psi^*\psi = 1$, show that $\vec{j} = q\vec{S}$ has units of charge per unit area per unit time, which are the dimensions of current density.

(c) In part (a), you assumed that $\vec{\nabla}\cdot\vec{A} = 0$. Now show that $\vec{S}$ has *exactly the same form* in any gauge, *ie.* show that $\vec{S}$ is gauge invariant. That is, show that if we make the following transformations, then $\vec{S}'$ defined in terms of $\vec{A}'$ and $\psi'$ is identical to $\vec{S}$ defined in terms of $\vec{A}$ and $\psi$.

$$\begin{aligned}\vec{A}'(\vec{x},t) &= \vec{A}(\vec{x},t) - \vec{\nabla}f(\vec{x},t)\\ \psi'(\vec{x},t) &= \exp\left(-\frac{iq}{\hbar c}f(\vec{x},t)\right)\psi(\vec{x},t)\end{aligned}$$

where $f$ is any function of $\vec{x}$ and $t$.

[1] Note: it is always possible to find a gauge transformation that takes a given vector potential $\vec{A}(\vec{x})$ and turns it into one with $\vec{\nabla}\cdot\vec{A} = 0$. (Optional: show this.) Note that stating that $\vec{\nabla}\cdot\vec{A} = 0$ does not fully specify $\vec{A}$. For example, the magnetic field $\vec{B} = (0, 0, B_0)$ can be described by $\vec{A} = (-B_0 y, 0, 0)$ or $\vec{A} = (0, B_0 x, 0)$, both of which satisfy $\vec{\nabla}\cdot\vec{A} = 0$.

3. **Translation Invariance in a Uniform Magnetic Field (20 points)**

One of the surprising things in our analysis of the quantum mechanics of a particle in a uniform magnetic field is that even though $\vec{B}$ is uniform, and we would therefore expect translation invariance in the $xy$-plane, we find that, in any gauge we choose, the Hamiltonian does not appear to reflect this symmetry. This issue is explored in depth in the supplementary notes. In this problem, you explore it in a different gauge, and in a somewhat different way.

The resolution to this question is that translation operators which do commute with the Hamiltonian *can* be constructed. We shall see, however, that there is a catch.

Consider a magnetic field $\vec{B} = (0, 0, -B_0)$ and work in the gauge in which $\vec{A} = (B_0 y, 0, 0)$. The time-independent Schrödinger equation (for states in the $xy$-plane) is

$$\frac{-\hbar^2}{2m}\left[\frac{\partial^2\psi}{\partial y^2} + \left(\frac{\partial}{\partial x} - \frac{iqB_0}{\hbar c}y\right)^2\psi\right] = E\psi\ , \tag{6}$$

and the Hamiltonian is

$$H = \frac{1}{2m}\left[p_y^2 + \left(p_x - \frac{qB_0}{c}y\right)^2\right]\ . \tag{7}$$

(a) The appearance of $y$ destroys (on the face of it) invariance under translation in the $y$ direction. Show, however, that if $\psi(x, y)$ is a solution of (6), then so too is $\tilde{\psi}(x, y)$ defined by

$$\tilde{\psi}(x, y) = \psi(x, y - b)\exp(iqB_0 bx/\hbar c)\ . \tag{8}$$

[Hint: be careful with your notation. Express $(\partial/\partial x - iqB_0 y/\hbar c)\tilde{\psi}$ at the point $(x, y)$ in terms of $\psi$ and $\partial\psi/\partial x$ at the point $(x, y - b)$.]

(b) Consider the operator $V_b$ which I define by telling you how it acts on any state $|\psi\rangle$:

$$V_b|\psi\rangle = |\tilde{\psi}\rangle\ . \tag{9}$$

This operator clearly has the effect of translating in $y$ by a distance $b$. Show that $V_b$ is unitary, and show that it commutes with the Hamiltonian $H$. [Hint: this part of the problem is easy.]

(c) In parts (c) and (d), I ask you to find an explicit expression for $V_b$. You do not actually need this explicit expression for part (e), but having an explicit expression may make you feel more comfortable with $V_b$. Find an operator $Q$ which commutes with $H$ and generates translations in $y$. That is, you must find an operator which obeys $[Q, H] = 0$ and $[y, Q] = i\hbar$. [Hint: $Q$ should be a linear combination of the $p_y$ and $x$ operators.]

(d) Show that $V_b = \exp(-ibQ/\hbar)$. That is, show that this explicit expression for $V_b$ yields $V_b|\psi\rangle = |\tilde{\psi}\rangle$.

(e) In the gauge in which we are working, $x$ does not appear in the Hamiltonian. The translation operator for translation in the $x$-direction is therefore the standard one. Call the operator which translates by $a$ in the $x$-direction $U_a$. That is,

$$\langle x, y|U_a|\psi\rangle = \langle x - a, y|\psi\rangle \ . \tag{10}$$

[The explicit expression for $U_a$ is just $U_a = \exp(-iap_x/\hbar)$ but, again, you will not need this explicit expression.] You now have two translation operators, $U_a$ and $V_b$, both of which commute with $H$ for any values of $a$ and $b$ you like. So, what's the catch?

Calculate $\langle x, y|U_a V_b|\psi\rangle$ and $\langle x, y|V_b U_a|\psi\rangle$ and show that $U_a$ commutes with $V_b$ if and only if $ab$ is an integer multiple of $A_B = 2\pi\hbar c/qB_0$ and hence if and only if $abB_0$ is an integer multiple of $\Phi_0 = hc/q$.

4. **Counting the States in a Landau Level (15 points)**

Consider a charged particle in a magnetic field as in the previous problem. Work in the gauge chosen in the previous problem. The particle is restricted to move in a rectangular region of the $xy$-plane whose extent is $0 < x < a$ and $0 < y < b$.

From the result of the previous problem, if we choose $a$ and $b$ so that $abB_0 = N\Phi_0$, with $N$ some large positive integer, then we should be able to find states $|\psi\rangle$ which are simultaneous eigenstates of $H$, $U_a$ and $V_b$. In this problem we shall count all the states in the lowest Landau level by counting how many states $|\psi\rangle$ there are which satisfy $H|\psi\rangle = E_{LLL}|\psi\rangle$ and $U_a|\psi\rangle = |\psi\rangle$ and $V_b|\psi\rangle = |\psi\rangle$. Here, $E_{LLL} = \hbar eB_0/2mc$ is the energy of the lowest Landau level.

[Completely optional: In general, the eigenvalues of unitary operators are complex numbers of modulus one. You could therefore be wondering why the eigenvalues of $U_a$ and $V_b$ must be 1. If you wish, after you complete this problem you can go back and analyze what would have changed if we had required that the eigenvalues of $U_a$ and $V_b$ were $\exp(i\theta_U)$ and $\exp(i\theta_V)$ for real but nonzero $\theta_U$ and $\theta_V$. You will discover that the only change is in the nature of the boundary conditions satisfied by the wave function. As to the choices we have made, $\theta_U = 0$ corresponds to choosing periodic boundary conditions in the $x$-direction, whereas $\theta_V = 0$ corresponds to a different, particular, boundary condition in the $y$-direction.]

(a) Show that if $U_a|\psi\rangle = |\psi\rangle$, $V_b|\psi\rangle = |\psi\rangle$, and $\psi(x, y)$ satisfies periodic boundary conditions in the $x$-direction, then $\psi(x, y)$ must be of the form

$$\psi(x, y) = \sum_{n=-\infty}^{\infty} u_n(y) \exp(i2\pi nx/a) \tag{11}$$

with $u_{n+N}(y) = u_n(y - b)$. [Aside: the eigenstates I used when I counted states in lecture were eigenstates of $U_a$ but not of $V_b$. The eigenstates (11) will turn out to be linear combinations of those I used in lecture.]

(b) Use the Schrödinger equation $H|\psi\rangle = E_{LLL}|\psi\rangle$ (that is, the Schrödinger equation (6)) to show that

$$u_n(y) = c_n f(y + nb/N) \tag{12}$$

where the function $f(y)$ is the solution to the time-independent Schrödinger equation for a particle in the lowest energy state of a simple harmonic oscillator with frequency $\omega = eB_0/mc$.

(c) It might seem that you have found infinitely many solutions. $\psi$ is specified by an infinite set of constants $c_n$. If these constants can be chosen arbitrarily, then there would indeed be infinitely many linearly independent wave functions satisfying all the conditions. However, show that (a) and (b) imply that $c_{n+N} = c_n$. [Hint: don't forget that $abB_0 = N\Phi_0$.] This means that only $N$ of the $c_n$'s are independent. You have thus shown that there are are exactly $N$ states satisfying all the conditions. Thus, in a system with area $NA_B$ the lowest Landau level contains $N$ states.

[Optional: To complete the argument, you must check that you can find $N$ states which are orthogonal. To do this, construct $N$ states as follows: for each state, choose one out of $c_0 \ldots c_{N-1}$ to be 1, and the others to be zero. For example, the first of these states has $\ldots c_{-2N} = c_{-N} = c_0 = c_N = c_{2N} \ldots = 1$ and all other $c$'s zero. The second has $\ldots c_{-2N+1} = c_{-N+1} = c_1 = c_{N+1} = c_{2N+1} \ldots = 1$ and all other $c$'s zero. Etc. What you have shown above is that any state in the lowest landau level is a linear combination of these $N$ states. All you have to do now is show that these $N$ states are orthogonal. That is easy to do.]