

1. Brick in a Square Well

REMEMBER: THIS PROBLEM AND THOSE BELOW SHOULD NOT BE HANDED IN. THEY WILL NOT BE GRADED. THEY ARE INTENDED AS A STUDY GUIDE TO HELP YOU UNDERSTAND TIME DEPENDENT PERTURBATION THEORY AND THE GROVER ALGORITHM.

Here is a simple enough time dependent perturbation of a simple enough system that everything can be computed analytically.

Do Griffiths Problem 9.18. (Note: Problem 9.17 in 1st. Ed.)

2. A Time-Dependent Two-State System

Consider a two-state system with Hamiltonian

$$H(t) = \begin{pmatrix} +E & v(t) \\ v(t) & -E \end{pmatrix}$$

where $v(t)$ is real and where $v \rightarrow 0$ for $t \rightarrow \pm\infty$.

- (a) Suppose that at $t = -\infty$ the system is in the state $|1\rangle$. Use time dependent perturbation theory to determine the probability that at $t = +\infty$ the system is in the state $|2\rangle$, to lowest order in v .
- (b) If $E = 0$, the eigenstates of $H(t)$ do not depend on t . Use this fact to calculate the probability of a transition from $|1\rangle$ to $|2\rangle$ *exactly*, in this case. What is the result obtained from time-dependent perturbation theory in this case? What is the condition that the perturbative result is a good approximation to the exact result?

3. Excitation of a hydrogen atom

A hydrogen atom is placed in an electric field $\vec{E}(t)$ that is uniform and has the time dependence,

$$\begin{aligned} \vec{E}(t) &= 0 & t < 0 \\ &= \vec{E}_0 e^{-\gamma t} & t > 0 \end{aligned} \quad (5)$$

What is the probability that as $t \rightarrow \infty$, the hydrogen atom, initially in the ground state, makes a transition to the $2p$ state?

4. Decay of the three dimensional harmonic oscillator

The object of this problem is to calculate the lifetime of a charged particle (charge q , mass m) in the first p -state of the three dimensional harmonic oscillator (frequency ω).

- a) Write down an expression for the transition rate per unit time, $\Gamma(2p \rightarrow 1s)$, for the particle to *spontaneously* emit electromagnetic radiation and make a transition to the ground state. Γ should depend on the frequency of the emitted light and on the matrix element of the operator $q\vec{r}$.

Note that the $2p$ state is three-fold degenerate: it has $\ell = 1$ and can have $m_\ell = -1, 0, 1$.

- b) Show that the transition rate *is independent of* m_ℓ .
- c) Finally, give a formula for $\Gamma(2p \rightarrow 1s)$ in terms of m, ω, q , and fundamental constants.
- d) What is the relationship between the transition rate per unit time and the “lifetime” of the $2p$ state?

5. Lifetime of Excited States of Hydrogen

- (a) Do Griffiths Problem 9.11. (Problem 9.10 in 1st. Ed.) NOTE: You may use any results from Griffiths’ section 9.3.3 without proving them (even though Problem 9.10 comes before section 9.3.3). Using results from 9.3.3 without proving them turns this into a much easier problem.
- (b) Do Griffiths Problem 9.14, part (a) only. (Problem 9.13(a) in 1st. Ed.)

6. A wave front crossing a bound particle

Consider a particle in one dimension moving under the influence of some time-independent potential, $V(x)$. Assume that you know the energy levels and corresponding eigenfunctions for this problem. We now subject the particle to a traveling pulse represented by a space- and time-dependent potential,

$$V(t) = a\delta(x - ct)$$

where $\delta(x)$ is a Dirac δ -function.

- (a) Suppose as $t \rightarrow -\infty$ the particle is known to be in the ground state whose wavefunction is $\langle x|i \rangle = u_i(x)$. Find the probability for finding the system in some excited state, with wavefunction $\langle x|f \rangle = u_f(x)$ as $t \rightarrow \infty$.

- (b) (Note: you can skip part (b) and still do part (c). You may have to do this, depending on exactly what I am able to cover in lecture.) Reinterpret your result in part (a) as follows. Regard the δ -function pulse as a superposition of harmonic perturbations, by recalling that the δ function can be represented as a superposition of exponentials:

$$\delta(x - ct) = \frac{1}{2\pi c} \int_{-\infty}^{\infty} d\omega e^{i\omega(x/c-t)}. \quad (6)$$

Show that if you treat each frequency component of the δ function separately, using for each the result we obtained in lecture for a harmonic perturbation (namely that there is a transition if and only if $\omega = \omega_{fi}$ and the amplitude of that transition is the matrix element of the operator coefficient of the harmonic time dependence between the initial and final states) then you get the same result as in part (a).

The lesson is that the analysis we did in lecture with a harmonic time dependence can be applied to very different time dependences via Fourier transformation.

- (c) Apply the result of part (a) to the one dimensional (infinite) square well,

$$\begin{aligned} V(x) &= 0 && \text{for } 0 < x < d, \\ &= \infty && \text{for } x < 0 \text{ or } x > d \end{aligned} \quad (7)$$

Express the probability to transition from the ground state to the first excited state as a function of the dimensionless parameters $\alpha = \frac{a}{\hbar c}$ and $\beta = \frac{d\Delta E}{2\pi\hbar c}$, where $\Delta E = \frac{3\pi^2\hbar^2}{2md^2}$. Show that the transition probability has a maximum for $\beta \approx 1$. Explain this in terms of the time it takes light to cross the potential well and the natural timescale of the quantum system.

7. The Grover Algorithm

Consider the 8 dimensional Hilbert space formed by taking the tensor product of the Hilbert spaces for three spin-one-half particles.

We denote the basis states as follows:

$$\begin{aligned} |0\rangle &= |0, 0, 0\rangle \\ |1\rangle &= |0, 0, 1\rangle \\ |2\rangle &= |0, 1, 0\rangle \\ |3\rangle &= |0, 1, 1\rangle \\ |4\rangle &= |1, 0, 0\rangle \\ |5\rangle &= |1, 0, 1\rangle \\ |6\rangle &= |1, 1, 0\rangle \\ |7\rangle &= |1, 1, 1\rangle \end{aligned}$$

where, for example, $|0, 1, 0\rangle$ means a state in which all three spins are in eigenstates of S_z , with eigenvalues $+\hbar/2, -\hbar/2, +\hbar/2$.

Throughout this problem you will be constructing a variety of 8×8 matrices, working in a basis with basis vectors ordered as above.

- (a) The first stage of the Grover algorithm is initialization. Suppose we start with all spins up, namely in state $|0\rangle$. We want to find a unitary operator $U_{\text{initialize}}$ such that

$$U_{\text{initialize}}|0\rangle = |s\rangle$$

where the state $|s\rangle$ is given by

$$|s\rangle = \frac{1}{\sqrt{8}} \left[|0\rangle + |1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle + |7\rangle \right].$$

Construct the 8×8 matrix $U_{\text{initialize}}$ as the product of three 8×8 unitary matrices each of which acts only within the Hilbert space of one of the three spins.

Note: my guess is that this is the part of this problem that you will find trickiest. Note that you need not do this part of the problem in order to do any of the other parts.

- (b) Lets suppose that $f(3) = 1$ and $f(a) = 0$ for $a = 0, 1, 2, 4, 5, 6, 7$. In other words, “3 is the winner”. Define a diagonal unitary matrix called $(-1)^f$ that acts on basis states as follows:

$$\begin{aligned} (-1)^f|a\rangle &= |a\rangle \text{ for } a \neq 3 \\ (-1)^f|3\rangle &= -|3\rangle. \end{aligned}$$

Write $(-1)^f$ as an 8×8 matrix.

Note: this part of the problem is very easy as posed. Too easy, in fact. Doing it this way is a little too much like “looking inside the black box and seeing how f works”. What you should really do is construct this unitary operator by introducing a “work-bit”, introducing an operator U_f which represents a function call via $U_f|a, 0\rangle = |a, f(a)\rangle$ and $U_f|a, 1\rangle = |a, 1 - f(a)\rangle$, introducing the operator L defined in lecture, and then constructing $(-1)^f = U_f L U_f$. I do recommend that you do this explicitly, but adding the work bit means doubling the Hilbert space to 16×16 so I am not going to ask you to turn this in.

- (c) Write the unitary operator $U_s \equiv 2|s\rangle\langle s| - 1$ as an 8×8 matrix. (You should check that your matrix is unitary, but do not turn this check in.)

(d) Find the state

$$\left[U_s (-1)^f \right]^k |s\rangle$$

for $k = 0, 1, 2, 3$. You should find that for $k = 2$, it is fairly close to the state $|3\rangle$ while for $k = 3$, it has become less close to $|3\rangle$.

Suppose that the state with $k = 2$ “is measured”, meaning that S_z is measured for each of the three spins. What is the probability that the outcome of this measurement will be $+\hbar, -\hbar, -\hbar$ (which corresponds to the state $|3\rangle$)? That is, what is the probability that upon measurement you get the right answer?

Note: I proved in lecture that for large N , the best choice for k is the integer closest to $\pi\sqrt{N}/4$. For our $N = 8$, which is not even very large, $\pi\sqrt{N}/4 = 2.22$. Now that you have understood the $N = 8$ example explicitly, you should review the proof of the large- N result. [Note: although it is not really necessary, it is fine if you choose to use a program like Mathematica to multiply out matrices.]