PROFESSOR: Last time we spoke about the Stern-Gerlach experiment, and how you could have a sequence of Stern-Gerlach boxes that allow you to understand the type of states and properties of the physics having to do with spin-1/2. So the key thing in the Stern-Gerlach machine was that a beam of silver atoms, each of which is really like an electron with a magnetic moment, was placed in an inhomogeneous strong magnetic field, and that would classically mean that you would get a deflection proportional to the z-component of the magnetic moment.

What was a surprise was that by the time you put the screen on the right side, it really split into two different beams, as if the magnetic moments could either be all the way pointing in the z-direction or all the way down, pointing opposite to the z-direction, and nothing in between. A very surprising result.

So after looking at a few of those boxes, we decided that we would try to model the spin-1/2 particle as a two-dimensional complex vector space. What is the two-dimensional complex vector space? It's the possible space of states of a spin-1/2 particle.

So our task today to go into detail into that, and set up the whole machinery of spin-1/2. So we will do so, even though we haven't quite yet discussed all the important concepts of linear algebra that we're going to need. So today, I'm going to assume that at least you have some vague notions of linear algebra reasonably well understood. And if you don't, well, take them on faith today. We're going to go through them slowly in the next couple of lectures, and then as you will reread this material, it will make more sense.

So what did we have? We said that the spin states, or the possible states of this silver atom, that really correspond to an election, could be described by states z comma plus and z colon minus So these are the two states.

This state we say corresponds to an angular momentum Sz hat. Sz-- I can put it like that-- of h-bar over 2, and this corresponds to Sz equals minus h-bar over 2. And those are our two states. The z label indicates that we've passed, presumably, these atoms through a filter in the z-direction, so that we know for certain we're talking about the z-component of angular momentum of this state. It is positive, and the values here again have the label z to remind us that we're talking about states that have been organized using the z-component of angular momentum.

You could ask whether this state has some angular momentum-- spin angular momentum-- in the x-direction or in the y-direction, and we will be able to answer that question in an hour from now.

So mathematically, we say that this statement, that this state, has Sz equals h-bar over 2 means that there is an operator, Sz hat-- hat for operators. And this operator, we say, acts on this state to give h-bar over 2 times this state.

So when we have a measurement in quantum mechanics, we end up talking about operators. So this case is no exception. We think of the operator, Sz, that acts in this state and gives h-bar over 2. And that same operator, Sz, acts on the other state and gives you minus h-bar over 2 times the state.

You see, an operator on a state must give a state. So in this equation, we have a state on the right, and the nice thing is that the same state appears on the right. When that happens, you say that the state is an eigenstate of the operator. And, therefore, the states z plus, minus are eigenstates of the operator Sz with eigenvalues-- the number that appears here-- equal to plus, minus h over 2.

So the relevant physical assumption here is the following, that these two states, in a sense, suffice. Now, what does that mean? We could do the experiment again with some Stern-Gerlach machine that is along the x-axis, and say, oh, now we've got states x plus and x minus and we should add them there. They are also part of the possible states of the system. Kind of. They are parts of the possible states of the system. They are possible states of the system, but we shouldn't add them to this one.

These will be thought as basis states. Just like any vector is the superposition of a number times the x-unit vector plus a number times the y-unit vector and a number times the z-unit vector, we are going to postulate, or try to construct the theory of spin, based on the idea that all possible spin states of an electron are obtained by suitable linear superposition of these two vectors. So, , in fact, what we're going to say is that these two vectors are the basis of a two-dimensional vector space, such that every possible state is a linear superposition.

So psi, being any possible spin state, can be written as some constant, C1 times z plus plus C2 times z minus where these constants, C1 and C2 belong to the complex numbers.

And by this, we mean that if any possible state is a superposition like that, the set of all possible states are the general vectors in a two-dimensional complex vector space. Complex vector space, because the coefficients are complex, and two-dimensional, because there's two basis vectors.

Now this doesn't quite look like a vector. It looks like those things called kets. But kets are really vectors, and we're going to make the correspondence very clear.

So this can be called the first basis state and the second basis state. And I want you to realize that the fact that we're talking about the complex vector space really means these coefficients are complex. There's no claim that the vector is complex in any sense, or this one. They're just vectors. This is a vector, and it's not that we say, oh this vector is complex. No. A complex vector space, we think of as a set of vectors, and then we're allowed to multiply them by complex numbers.

OK, so we have this, and this way of thinking of the vectors is quite all right. But we want to be more concrete. For that, we're going to use what is called a representation. So I will use the word representation to mean some way of exhibiting a vector or state in a more concrete way. As something that any one of us would call a vector.

So as a matter of notation, this being the first basis state is sometimes written as a ket with a 1. Like that. And this being this second basis state is sometimes written this way. But here is the real issue of what we were calling a representation. If this is a two-dimensional vector space, you're accustomed to three-dimensional vector space. What are vectors? They're triplets of numbers. Three numbers. That's a vector.

Column vectors, it's perhaps easier to think about them. So column vectors. So here's what we're going to say. We have this state z plus. It's also called 1. It's just a name, but we're going to represent it as a column vector. And as a column vector, I'm going to represent it as the column vector 1, 0.

And this is why I put this double arrow. I'm not saying it's the same thing-- although the really it is-- it's just a way of thinking about it as some vector in what we would call canonically a vector space. Yes

AUDIENCE: So do the components of the column vector there have any correspondence to the actual. Does it have any basis in the actual physical process going on? Or, what is their connection to the actual physical [INAUDIBLE] represented here?

PROFESSOR: Well, we'll see it in a second. It will become a little clearer. But this is like saying, I have a two-dimensional two0 vector space, so I'm going to think of the first state as this vector. But how do I write this vector? Well, it's the vector ex. Well, if I would write them in components, I would say, for a vector, I can put two numbers here, a and b. And this is the a-component and b-component. So here it is, ex would be 1, 0. And ey would be 0, 1. If I have this notation then the point a, b is represented by a and b as a column vector.

So at this moment, it's just a way of associating a vector in the two-dimensional canonical vector space. It's just the column here.

So the other state, minus-- it's also called 2-- will be represented by 0 1 1. 1 And therefore, this state, psi, which is C1 z plus plus C2 z minus will be represented as C1 times the first vector plus C2 times the second vector. Or multiplying, in C1, C2.

So this state can be written as a linear superposition of these two basis vectors in this way-- you can write it this way. You want to save some writing, then you can write them with 1 and 2. But as a vector, it's represented by a column vector with two components. That's our state.

Now in doing this, I want to emphasize, we're introducing the physical assumption that this will be enough to describe all possible spin states, which is far from obvious at this stage. Nevertheless, let's use some of the ideas from the experiment, the Stern-Gerlach experiment.

We did one example of a box that filtered the plus z states, and then put it against another z machine, and then all the states went through the up. Which is to say that plus states have no amplitude, no probability to be in the minus states. They all went through the plus. So when we're going to introduce now the physical translation of this fact, as saying that these states are orthogonal to each other.

So, this will require the whole framework, in detail, of bras and kets to say really, precisely-- but we're going to do that now and explain the minimum necessary for you to understand it. But we'll come back to it later. So this physical statement will be stated as z minus with z plus. The overlap, the bra-ket of this, is 0. The fact that all particles went through and went out through the plus output will state to us, well, these states are well normalized. So z plus, z plus is 1

Similarly, you could have blocked the other input, and you would have concluded that the minus state is orthogonal to the plus. So we also say that these, too, are orthogonal, and the minus states are well normalized.

Now here we had to write four equations. And the notation, one and two becomes handy, because we can summarize all these statements by the equation Ij equals delta Ij. Look, if this equation is 2 with 1 equals 0. The bra 2, the ket 1. This is 1 with 1 is equal to 1. Here is is 1 with 2 is equal to 0, and 2 with 2 is equal to 1. So this is exactly what we have here.

Now, I didn't define for you these so-called bras. So by completeness, I will define them now. And the way I will define them is as follows. I will say that while for the one vector basis state you associate at 1, 0, you will associate to one bra, the row vector 1, 0. I sometimes tend to write equal, but-- equal is all right-- but it's a little clearer to say that there's arrows here.

So we're going to associate to 1, 1, 0-- we did it before-- but now to the bra, we think of the rho vector. Like this. Similarly, I can do the same with 2. 2 was the vector 0, 1. It's a column vector, so 2 was a bra. We will think of it as the row vector 0, 1.

We're going to do this now a little more generally. So, suppose you have state, alpha, which is alpha 1, 1 plus alpha 2, 2. Well, to this, you would associate the column vector alpha 1, alpha 2. Suppose you have a beta state, beta 1, 1 plus beta 2, 2. You would associate beta 1, beta 2 as their representations.

Now here comes the definition for which this is just a special case. And it's a definition of the general bra. So the general bra here, alpha, is defined to be alpha 1*, bra of the first, plus alpha 2*, bra of the second. So this is alpha 1* times the first bra, which we think of it as 1, 0, plus alpha 2* times the second bra, which is 0, 1. So this whole thing is represented by alpha 1*, alpha 2*.

So, here we've had a column vector representation of a state, and the bra is the row vector representation of the state in which this is constructed with complex conjugation. Now these kind of definitions will be discussed in more detail and more axiomatically early very soon, so that you see where you're going. But the intuition that you're going to get from this is quite valuable.

So what is the bra-ket? Alpha-beta is the so-called bra-ket. And this is a number. And the reason for complex conjugation is, ultimately, that when these two things are the same, it should give a positive number. It's like the length squared. So that's the reason for complex conjugation, eventually.

But, for now, you are supposed to get a number from here. And the a reasonable way to get a number, which is a definition, is that you get a number by a matrix multiplication of the representatives. So you take the representative of alpha, which is alpha 1*, alpha 2*. And do the matrix product with the representative of beta, which is beta 1, beta 2. And that's alpha 1*, beta 1 plus alpha 2*, beta 2. And that's the number called the inner product, or bra-ket product. And this is the true meaning of relations of this kind. If you're given an arbitrary states, you compute the inner product this way. And vectors that satisfy this are called orthonormal because they're orthogonal and normal with respect to each other in the sense of the bra and ket.

So this definition, as you can see, is also consistent with what you have up there, and you can check it. If you take I with j, 1, say, with 2-- like this-- you do the inner product, and you get 0. And similarly for all the other states.

So let's then complete the issue of representations. We had representations of the states as column vectors-- two by two column vectors or row vectors. Now let's talk about this operator we started with. If this is an operator, acting on states, now I want to think of its representation, which would be the way it acts on these two component vectors. So it must be a two by two matrix, because only a two by two matrix acts naturally on two component vectors.

So here is the claim that we have. Claim, that Sz hat is represented-- but we'll just put equal-- by this matrix.

You see, it was an operator. We never talked about matrices. But once we start talking about the basis vectors as column vectors, then you can ask if this is correct.

So for example, I'm supposed to find that Sz hat acting on this state 1 is supposed to be h-bar over 2 times the state 1. You see? True.

Then you say, oh let's put the representation, h-bar over 2, 1 minus 1, 0, 0. State one, what's its representation? 1, 0.

OK, let's act on it. So, this gives me h-bar over 2. I do the first product, I get a 1. I do the second product, I get a 0. Oh, that seems right, because this is h over 2 times the representation of the state 1.

And if I check this, and as well that Sz on 2 is equal minus h-bar over 2, 2-- which can also be checked-- I need to check no more. Because it suffices that this operator does what it's supposed to do of the basis vectors. And it will do what it's supposed to do on arbitrary vectors.

So we're done. This is the operator Sx, and we seem to have put together a lot of the ideas of the experiment into a mathematical framework.

But we're not through because we have this question, so what if you align and operate the machine along x? What are the possible spin states along the x-direction? How do you know that all that the spins state that points along x can be described in this vector space? How do I know there exists a number C1, C2 so that this linear combination is a spin state that points along x.

Well, at this moment, you really have to invent something. And the process of invention is never a very linear one. You use analogies-- you use whatever you can-- to invent what you need. So, given that that's a possibility, we could follow what Feynman does in his Feynman lectures, of discussing how to begin rotating Stern-Gerlach machines, and doing all kinds of things. It's an interesting argument, and it's a little hard to follow, a little tedious at points. And we're going to follow a different route.

I'm going to assume that you remember a little about angular momentum, and I think you do remember this much. I want to say, well, this is spin angular momentum. Well, let's compare it with orbital angular momentum, and see where we are.

You see, another way of asking the question would be, well, what are the operators Sx and Sy. Where do I get them? Well, the reason I want to bring in the angular momentum is because there you have Lz, but you also have Lx and Ly. So angular momentum had Lz, just like we had here, but also Lx and Ly.

Now these spin things look a lot more mysterious, a lot more basic, because, like Lz, it was xpy minus ypx. So you knew how this operator acts on wave functions. You know, it multiplies by y, takes an x derivative, or it's a dd phi.

It has a nice thing, but Sz on the other hand, there's no x, there's no derivatives. It's a different space. It's working in a totally different space, in the space of a two-dimensional complex vector space of column vectors with two numbers. That's where it acts. I'm sorry there's no dd x, nothing familiar about it. But that's what we have been handed.

So this thing acts on wave functions, and thus natural things. Well, the other one acts on column vectors. Two-by-two-- two component column vectors, and that's all right. But we also know that Lz is Hermitian. And that was good, because it actually meant that this is good observable. You can measure it.

Is Sz Hermitian? Well, yes it is. Hermeticity of a matrix-- as we'll discuss it in a lot of detail, maybe more than you want-- means you can transpose it complex conjugated, and you get the same matrix.

Well that matrix is Hermitian. So that's nice. That maybe is important. So what other operators do we have? Lx and Ly. And if we think of Lx as L1, Ly as L2, and Lz as L3, you had a basic computation relation. Li with Lj was equal to i epsilon ijk Lk-hat-- oops i-hbar.

And this was called the algebra of angular momentum. These three operators satisfy these identities. i and j are here, k is supposed to be summed over-- repeated in this is our sum from 1, 2, and 3. And epsilon ijk is totally anti-symmetric with epsilon 1, 2, 3 equal to plus 1.

You may or may not know this epsilon. You will get some practice on that very soon. Now for all intents and purposes, we might as well write the explicit formulas between Lx, Ly equal i-hbar Lz. Ly Lz equals i-hbar Lx. And Lz Lx-- there are hats all over-- equal i-hbar Ly.

So we had this for orbital angular momentum, or for angular momentum in general. So what we're going to do now is we're going to try to figure out what are Sx and Sy by trying to find a complete analogy. We're going to declare that S is going to be angular momentum. So we're going to want that Sx with Sy will be i-hbar Sz. Sy with Sz will be i-hbar Sx. And finally, Sz with Sx is i-hbar Sy.

And we're going to try that these things be Hermitian. Sx and Sy.

So let me break for a second and ask if there are questions. We're aiming to complete the theory by taking S to be angular momentum, and see what we get. Can we invent operators Sx and Sy that will do the right thing? Yes.

AUDIENCE: What's the name for the epsilon ijk? I know there's a special name for the [INAUDIBLE].

PROFESSOR: What's the name?

AUDIENCE: There's a [INAUDIBLE] tensor.

PROFESSOR: That's right. Let [INAUDIBLE] to be the tensor. It can be used for cross products. It's very useful for cross products. It's a really useful tensor. Other questions. More questions about what we're going to try to do, or this so far. Yes.

AUDIENCE: When you use the term representation, is that like the technical mathematical term of representation, like in algebra?

PROFESSOR: Yes. It's representation of operators in vector spaces. So we've used the canonical vector space with column vectors represented by entries one and numbers. And then the operators become matrices, so whenever an operator is viewed as a matrix, we think of it as a representation. Other questions. Yes.

AUDIENCE: Will we talk about later why we can make an analogy between L and S? Or is it [INAUDIBLE]?

PROFESSOR: Well you see, this is a very strong analogy, but there will be big differences from orbital angular momentum and spin angular momentum. And basically having to do with the fact that the eigenvalues of these operators are plus minus h-bar over 2. And in the orbital case they tend to be plus minus integer values of h-bar.

So this is a very deep statement about the algebra of these operators that still allows the physics of them to be quite different. But this is probably the only algebra that makes sense. It's angular momentum. So we're going to try to develop that algebra like that, as well here. You could take it to be an assumption.

And as I said, an experiment doesn't tell you the unique way to invent the mathematics. You try to invent the consistent mathematics and see if it coincides with the experiment. And this is a very natural thing to try to invent

So what are we facing? We're facing a slightly nontrivial problem of figuring out these operators. And they should be Hermitian. So let's try to think of Hermitian two-by-two matrices.

So here is a Hermitian two-by-two matrix. I can put an arbitrary constant here because it should be invariant on their transposition, which doesn't change this diagonal value in complex conjugation. So c should be real. d should be real. For the matrix to be Hermitian, two-by-two matrix, I could put an a here. And then this a would have to appear here as well. I can put minus ib, and then I would have plus ib here.

So when I transpose a complex conjugate, I get this one. So this matrix with abc and d real is Hermitian.

Hermiticity is some sort of reality condition. Now, for convenience, I would put a 2c and a 2d here. It doesn't change things too much.

Now to look at what we're talking about. We're talking about this set of Hermitian matrices. Funnily, you can think of that again as a vector space. Why a vector space? Well, we'll think about it, and in a few seconds, it will become clear. But let me just try to do something here that might help us.

We're trying to identify Sx and Sy from here so that this commutation relations hold.

Well, if Sx and Sy have anything to do with the identity matrix, they would commute with everything and would do nothing for you.

So, I will remove from this matrices then trying to understand something having to do with the identity. So I'll remove a Hermitian matrix, which is c plus d times the identity-- the two-by-two identity matrix. This is a Hermitian matrix, as well. And I can remove it, and then this matrix is still Hermitian, and this piece that I've removed doesn't change commutators as they appear on the left hand side. So if you have an Sx and an xy here, and you're trying to do a computation, it would not contribute, so you might as well just get rid of them.

So if we remove this, we are left with-- you're subtracting c plus d from the diagonal. So here you'll have c minus d. Here you'll get b minus c, a minus ib, and a plus ib. And we should keep searching for Sx and Sy among these matrices.

But then you say, look, I already got Sz, and that was Hermitian. And Sz was Hermitian, and it had a number, and the opposite number on the other diagonal entry. If Sx and Sy have a little bit of Sz, I don't care. I don't want these to be independent matrices. I don't want to confuse the situation.

So if this thing has something along Sz, I want it out. So since precisely this number is opposite to this one, I can add to this matrix some multiple of Sz and kill these things in the diagonal. So add the multiple and Sz multiple, and we finally get this matrix. 0, a minus ib, a plus ib, and 0.

So we've made quite some progress. Let's see now what we have. Well, that matrix could be written as a times 0, 1, 1, 0 plus b times 0, minus i, i, 0.

Which is to say that it's this Hermitian matrix times a real number, and this Hermitian matrix times a real number. And that makes sense because if you take a Hermitian matrix and multiply by a real number, the matrix is still Hermitian. So this is still Hermitian because these are real. This is still Hermitian because a is real, and if you add Hermitian matrices, it's still Hermitian.

So in some sense, the set of Hermitian matrices, two-by-two Hermitian matrices, is a real vector space with four basis vectors. One basis vector is this, another basis vector is this, the third basis vector is the Sz part, and the fourth basis vector is the identity that we subtracted. And I'm listing the other two that we got rid of because physically we're not that interested given that we want Sx and Sz. So, Sx and Sy.

But here it is. These four two-by-two matrices are sort of the linearly independent Hermitian matrices. You can think of them as vectors, four basis vectors. You multiply by real numbers, and now you add them, and you got the most general Hermitian matrix. So this is part of the subtlety of this whole idea of vector spaces of matrices, which can be thought of as vectors sometimes, as well.

So that's why these matrices are quite famous. But before we just discuss why they are so famous, let's think of this. Where we're looking for Sx and Sy, and we actually seem to have two matrices here that could do the job, as two independent Hermitian two-by-two matrices.

But we must add a little extra information. We don't know what the scale is. Should I multiply this by 5 and call that Sx? Or this by 3? We're missing a little more physics. What is the physics? The eigenvalues of Sx should also be plus minus h over 2. And the eigenvalues of Sy should also be plus minus h over 2. Just like for Sz. you could have started the whole Stern-Gerlach things thinking of x, and you would have obtained plus minus h over 2.

So that is the physical constraint. I have to figure out those numbers. Maybe Sx is this one, as y is this one. And you can say, oh, you never told us if you're going to get the unique answer here. And yes, I did tell you, and you're not going to get a unique answer. There are some sign notations and some other things, but any answer is perfectly good.

So once you get an answer, it's perfectly good. Of course, we're going to get the answer that everybody likes. And the convention is that happily that everybody uses this same convention. Questions.

AUDIENCE: So I have a related question, because at the beginning we could have chosen the top right entry to be a plus ib and the bottom left to be a minus ib and that would have yielded a different basis matrix.

PROFESSOR: Right, I would have called this plus and minus. Yes.

AUDIENCE: Are we going to show that this is the correct form?

PROFESSOR: No, it's not the correct form. It is a correct form, and it's equivalent to any other form you could find. That's what we can show. In fact, I will show that there's an obvious ambiguity here. Well, in fact, maybe I can tell it do you, I think.

If you let Sx go to minus Sy, and Sy goes to plus Sx, nothing changes in these equations. They become the same equations. You know, Sx would become minus Sy, and this Sx-- this is not changed. But, in fact, if you put minus Sy and Sx as the same commutator then this one will become actually this commutator, and this one will become that. So I could change whatever I get for Sx, change it from minus Sy, for example, and get the same thing.

So there are many changes you can do. The only thing we need is one answer that works. And I'm going to write, of course, the one that everybody likes. But don't worry about that.

So let's think of eigenvectors and eigenvalues now. I don't know how much you remember that, but we'll just take it at this moment that you do.

So 0, 1, 1, 0 has two eigenvalues, and lambda equals 1, with eigenvector 1 over square root of 2, 1, 1. And a lambda equals minus 1 with eigenvector 1 over square root of 2, 1, minus 1.

The other one, it's equally easy to do. We'll discuss eigenvectors and eigenvalues later. Minus i, i, 0, 0, plus a lambda equals one eigenvector, with components 1 over square root of 2, 1, and i. I'm pretty sure it's 1 and i. Yes, and a lambda equals minus 1, with components 1 over square root 2, 1, minus i.

Now I put the 1 over square root of 2 because I wanted them to be normalized. Remember how you're supposed to normalize these things. You're supposed to take the row vector, complex conjugate, and multiply. Well, you would get 1 for the length of this, 1 for the length of this. You would get one for the length of this, but remember, you have to complex conjugate, otherwise you'll get 0. Also, you will get one for the length of this.

So these are our eigenvalues. So actually, with eigenvalues lambda equals 1 and minus 1 for these two, we're in pretty good shape. We could try Sx to be h-bar over 2 times 0, 1, 1, 0. And Sy to be h-bar over 2, 0, minus i, i, 0.

These would have the right eigenvalues because if you multiply a matrix by a number, the eigenvalue gets multiplied by this number, so the plus minus 1s become plus minus h-bar over 2.

But what are we was supposed to check? If this is to work, we're supposed to check these commutators. So let's do one, at least. Sx commutator with Sy. So what do we get? h-bar over 2, h-bar over 2-- two of them-- then the first matrix, 0, 1, 1, 0 times 0, minus i, i, 0, minus 0, minus i, i, 0 times 0, 1, 1, 0. Which is h-bar over 2 times h-bar over 2, times i, 0, 0, minus i, minus minus i, 0, 0, i. And we're almost there.

What do we have? Well, we have h-bar over 2, h-bar over 2. And we've got 2i and minus 2i. So this is h-bar over 2 times h-bar over 2, times 2i times 1, minus 1. And this whole thing is i h-bar, and the other part is h-bar over 2, 1, minus 1, which is i h-bar as z-hat.

Good, it works. You know, the only thing that could have gone wrong-- you could have identified 1 with a minus, or something like that It would have been equally good. Once you have these operators, we're fine. So one has to check that the other ones work, and they do. I will leave them for you to check.

And therefore, we've got the three matrices. It's a very important result-- the Sx Sy, and Sz. I will not rewrite them, but they should be boxed nicely, the three of them together, with that one there on top of the blackboard. And of course by construction, they're Hermitian.

They're famous enough that people have defined the following object. Si is defined to be h-bar over 2, sigma i, the power of the matrix sigmas. And these are poly matrices, sigma one is 0, 1, 1, 0. Sigma two is 0, minus i, i, 0. And sigma three is equal to 1, minus 1, 0, 0.

OK, so in principle-- yes, question.

AUDIENCE: Is it at all significant that the poly matrices are all squared [INAUDIBLE]?

PROFESSOR: Yes, it is significant. We'll use it, but at this moment, it's not urgent for us. We'll have no application of that property for a little while, but it will help us do a lot of the algebra of the poly matrices.

AUDIENCE: [INAUDIBLE] eigenvalues, right?

PROFESSOR: Sorry?

AUDIENCE: Doesn't that follow from the eigenvalue properties that we've [INAUDIBLE] plus or minus one. Because those were both squared. [INAUDIBLE].

PROFESSOR: That's right. I think so. Our eigenvalues-- yes, it's true. That the fact that the eigenvalues are plus minus 1 will imply that these matrices squared themselves. So it's incorporated into our analysis. The thing that I will say is that you don't need it in the expression of the commutators. So in the commentators, it didn't play a role to begin with. Put it as an extra condition.

Now what is the next thing we really want to understand? Is that in terms of plain states, we now have the answer for most of the experiments we could do. So in particular, remember that we said that we would have Sx, for example, having states x plus minus, which are h-bar over 2 plus minus, x comma plus minus. The states along the x-direction referred like that would be the eigenstates of the Sx operator. But we've calculated the states of the Sx operator-- they're here. The Sx operator is h-bar over 2 times this matrix. And we have those things. So the plus eigenvalue and the minus eigenvalue will just show up here.

So let me write them, and explain, in plain language, what these states are. So the eigenstate with lambda equal 1-- that would correspond to h-bar over two-- so the x plus corresponds to this vector. So what is that state? It's that vector which, if you want more explicitly, it's the z plus, plus z minus. This is the state 1 over square root of 2, 1, 1. The x minus is z plus, minus z minus. As you see on that blackboard, it's 1 minus 1.

So here it is. The states that you were looking for, that are aligned along x-- plus x or minus x-- are not new states that have you to add to the state space. They are linear combinations of the states you've got.

We can invert this formula and write, for example, that z plus is 1 over square root of 2, x plus, plus 1 over square root of 2, x minus. And z minus is 1 over square root of 2, x plus, minus 1 over square root of-- minus the square root of 2 is already out, I'm sorry-- minus x minus.

So actually, this answers the question that you had. For example, you put a z plus state, and you put an x filter-- what amplitude do you have to find a state in the x plus, given that you start with a state on the z plus?

Well, you put an x plus from here. You get 1 from this and 0 from this one because the states are always orthogonal. The states are orthogonal-- you should check that. And therefore, this is 1 over square root of 2. If you ask for x minus with respect to z plus, that's also 1 over square root of 2. And these are the amplitudes for this state to be found in this, for this state to be found in them. They're equal. The probabilities are 1/2.

And that's good. Our whole theory of angular momentum has given us something that is perfectly consistent with the Stern-Gerlach experiment, and it gives you these probabilities.

You can construct in the same way the y states. So the y states are the eigenstates of that second matrix, Sy, that we wrote on the left. So this matrix is Sy, so its eigenstates-- I'm sorry, Sy is there. Sy is there. The eigenstates are those, so immediately you translate that to say that Sy has eigenstates y plus minus, whose eigenvalues are plus minus h-bar over 2y plus minus. And y plus is equal 1 over square root of 2, z plus-- and look at the first eigenvector-- plus iz minus. And, in fact, they can put one formula for both. Here they are.

So, it's kind of neat that the x1s were found by linear combinations, and they're orthogonal. Now, if you didn't have complex numbers, you could not form another linear combination of this orthogonal. But thanks to these complex numbers, you can put an i there-- there's no i in the x ones-- and the states are orthogonal, something that you should check.

So again, you can invert and find the z states in terms of y, and you would conclude that the amplitudes are really the same up to signs, or maybe complex numbers, but the probabilities are identical.

So we've gotten a long way. We basically have a theory that seems to describe the whole result of the Stern-Gerlach experiment, but now your theory can do more for you. Now, in the last few minutes, we're going to calculate the states that are along arbitrary directions.

So here I produced a state that is along the x-direction plus, and along the x-direction minus. What I would like to construct, to finish this story, is a state that is along some arbitrary direction. So the state that points along some unit vector n.

So here is space, and here's a unit vector n with components nx, , ny, and nz. Or you can write the vector n as nx ex plus ny ey plus nz ez. And I would like to understand how I can construct, in general, a spin state that could be said to be in the n direction. We have the ones along the z, x, and y, but let's try to get something more general, the most general one.

So for this, we think of the triplet of operators S, which would be Sx, Sy, and Sz. Now you can, if you wish, write this as Sx-hat ex vector, plus Sy-hat ey vector, plus Sz hat ez vector. But this object, if you write it like that, is really a strange object. Think of it. It's matrices, or operators, multiplied by unit vectors. These vectors have nothing to do with the space in which the matrices act. The matrices act in an abstract, two-dimensional vector space, while these vectors are sort of for accounting purposes. That's why we sometimes don't write them, and say we have a triplet.

So this product means almost nothing. They're just sitting together. You could put the e to the left of the x or to the right. It's a vector. You're not supposed to put the vector inside the matrix, either. They don't talk to each. It's an accounting procedure. It is useful sometimes; we will use it to derive identities soon, but it's an accounting procedure.

So here's what I want to define. So this is a crazy thing, some sort of vector valued operator, or something like that. But what we really need is what we'll call S-hat n, which will be defined as n dot S. Where we take naively what a dot product is supposed to mean. This component times this component, which happens to be an operator. This times this, this times that. nx Sx plus ny Sy, plus nz Sz.

And this thing is something very intuitive. It is just an operator. It doesn't have anymore a vector with it. So it's a single operator. If your vector points in the z-direction, nx and ny z, and you have Sz because it's a unit vector. If the vector points in the x-direction, you get Sx. If the vector points in the y-direction, you get Sy. In general, this we call the spin operator in the direction of the vector n-- spin operator in the direction of n.

OK, so what about that spin operator? Well, it had eigenvalues plus minus h-bar over 2 along z, x, and y-- probably does still have those eigenvalues-- but we have to make this a little clearer.

So for that we'll take nx and ny and nz to be the polar coordinate things. So this vector is going to have a theta here on the azimuthal angle phi over here. So nz is cosine theta. nx and ny have sine theta. And nx cosine phi, and this one has sine phi.

So what is the operator Sn vector hat? Well, it's nx times Sx. So, I'll put a h-bar over 2 in front, so we'll have nx sigma x, or sigma1, plus ny sigma2, , plus nz sigma3.

Remember the spin operators are proportional h-bar over 2 times the sigmas-- so sigma1, sigma2, sigma3.

And look what we get. h-bar over 2. Sigma1 has an nx here, nx. Sigma2 has minus iny plus iny. And sigma3, we have a nz minus nz. So this is h-bar over 2, nz is cosine theta, nx minus iny-- you'd say, oh it's a pretty awful thing, but it's very simple-- nx minus iny is sine theta times e to the minus i phi. Here it would be sine theta, e to the i phi, and here we'll have minus cosine theta. So this is the whole matrix, Sn-hat, like that.

Well, in the last couple of minutes, let's calculate the eigenvectors and eigenvalues. So what do we get? Well, for the eigenvalues, remember what is the computation of an eigenvalue of a matrix. An eigenvalue for matrix a, you write that by solving the determinant of a minus lambda 1 equals 0.

So for any matrix a, if we want to find the eigenvalues of this matrix, we would have to write eigenvalues of Sn-hat. We have to ride the determinant of this, minus lambda i, so the determinant of h-bar over 2 cosine theta, minus lambda, minus h-bar over 2 cosine theta, minus lambda. And here, it's sine theta, e to the minus i phi, sine theta e to the i phi, the determinant of this being 0.

It's not as bad as it looks. It's actually pretty simple. These are a plus b, a minus b. Here the phases cancel out. The algebra you can read in the notes, but you do get lambda equals plus minus h-bar over 2.

Now that is fine, and we now want the eigenvectors. Those are more non-trivial, so they need a little more work. So what are you supposed to do to find an eigenvector? You're supposed to take this a minus lambda i, acting on a vector, and put it equal to zero. And that's the eigenvector.

So, for this case, we're going to try to find the eigenvector n plus. So this is the one that has Sn on this state-- well, I'll write it here, plus minus h over 2, n plus minus here. So let's try to find this one that corresponds to the eigenvalue equal to plus h-bar over 2.

Now this state is C1 times z plus, plus C2 times z minus. These are our basis states, so it's a little combination. Or it's C1, C2. Think of it as a matrix. So we want the eigenvalues of that-- the eigenvector for that-- so what do we have? Well, we would have Sn-hat minus h-bar over 2 times 1, on this C1, C2 equals 0. The eigenvector equation is that this operator minus the eigenvalue must give you that.

So the h-bars over 2, happily, go out, and you don't really need to worry about them anymore. And you get here cosine theta minus 1, sine theta e to the minus i phi, sine theta e to the i phi, and minus cosine theta minus 1, C1, C2 equals 0.

All right, so you have two equations, and both relate C1 and C2. Happily, and the reason this works is because with this eigenvalue that we've used that appears here, these two equations are the same. So you can take either one, and they must imply the same relation between C1 and C2. Something you can check.

So let me write one of them. C2 is equal to e to the i phi, 1 minus cosine theta over sine theta C1. It's from the first line.

So you have to remember, in order to simplify these things, your half angle identities. Sorry. 1 minus cosine theta is 2 sine squared theta over 2, and sine theta is 2 sine theta over 2 cosine theta over 2. So this becomes e to the i phi sine theta over 2, over cosine theta over 2, C1.

Now we want these things to be well normalized, so we want C1 squared plus C2 squared equal to 1. So, you know what C2 is, so this gives you C1 squared times 1 plus-- and C2 you use this, when you square the phase goes away-- sine squared theta over 2, cosine squared over 2 must be equal to 1. Well, the numerator is 1, so you learn that C1 squared is equal to cosine squared theta over 2.

Now you have to take the square root, and you could put an i or a phase or something. But look, whatever phase you choose, you could choose C1 to be cosine theta over 2, and say, I'm done. I want this one. Somebody would say, no let's put the phase, e so to the i pi over 5. So that doesn't look good, but four or even worse, this phase will show up in C2 because C2 is proportional to C1. So I can get rid of it.

I only should put it if I really need it, and I don't think I need it, so I won't put it. And you can always change your mind later-- nobody's going to take your word for this. So, in this case, C2 would be sine theta over 2, e to the i phi. It's nice, but it's [INAUDIBLE].

And therefore, we got this state n plus, which is supposed to be cosine theta over 2, z plus, and plus sine theta over 2, e to the i phi, z minus. This is a great result. It gives the arbitrarily located spin state that point in the n-direction. As a linear superposition of your two basis states, it answers conclusively the question that any spin state in your system can be represented in this two-dimensional vector space.

Now moreover, if I take that theta equals 0, I have the z-axis, and it's independent of the angle phi. The phi angle becomes singular at the North Pole, but that's all right. When theta is equal to 0, this term is 0 anyway. And therefore, this goes, and when theta is equal to 0, you recover the plus state.

Now you can calculate the minus state. And if you follow exactly the same economical procedure, you will get the following answer. And I think, unless you've done a lot of eigenvalue calculations, this is a calculation you should just redo.

So the thing that you get, without thinking much, is that n minus is equal to sine theta over 2 plus, minus cosine theta over 2, e to the i phi minus. At least some way of solving this equation gives you that. You could say, this is natural, and this is fine. But that is not so nice, actually.

Take theta equal to pi-- no, I'm sorry. Again, you take theta equal to 0. Theta equal to 0-- this is supposed to be the minus state along the direction. So this is supposed to give you the minus state. Because the vector n is along up, in the z-direction, but you're looking at the minus component. So theta equals 0.

Sure, there's no plus, but theta equals 0, and you get the minus state. And this is 1, and phi is ill-defined-- it's not so nice, therefore-- so, at this moment, it's convenient to multiply this state by e to the minus i phi, times minus 1. Just multiply it by that, so that n minus is equal to minus sine theta over 2, e to the minus i phi, plus, plus cosine theta over 2, minus. And that's a nice definition of the state.

When theta is equal to 0, you're fine, and it's more naturally equivalent to what you know. Theta equal to 0 gives you the minus state, or z minus. I didn't put the zs here, for laziness. And for theta equal to 0, the way the phase phi doesn't matter. So it's a little nicer. You could work with this one, but you might this well leave it like that.

So we have our general states, we've done everything here that required some linear algebra without doing a review of linear algebra, but that's what we'll start to do next time.