PROFESSOR: All right. So today, we'll continue our kind of review that included, of course, the last lecture, the variational principle that's supposed to be new stuff you didn't see in 804. And today, as we continue, we'll talk about position and momentum for about 30 minutes or 40 minutes, and then begin the study of spin. That will be spin-1/2 with a Stern-Gerlach experiment and the mathematics that comes out of it.

Now, we will talk about the Stern-Gerlach experiment in quite some detail so that you can appreciate what was going on there. And then we will extract a few of the mathematical lessons that this experiment tells us about quantum mechanics. Immediately after that, which will be probably middle of next lecture, we will pivot.

And as we learn this mathematics that the Stern-Gerlach experiment is telling us or asking us for, we will go in some detail on the necessary mathematics for quantum mechanics. We'll talk about vector spaces, linear operators, Hermitian operators, unitary operators, [INAUDIBLE], matrix representations, all kinds of things. That probably will be about two weeks, three lectures at least.

So it will be a nice study. And in that way, people that don't have a background in linear algebra will feel more comfortable with what we're going to be doing. And I think even for the people that have a background in linear algebra, you will gain a new appreciation about the concepts that we meet here.

So today, we begin, therefore, with position and momentum, and these are operators in quantum mechanics. And they have letters to denote them. x, we put a hat with it, that's a position operator. p, we put a hat on it. And the position and momentum operators don't commute. And the commutator is given by ih bar.

Now, we have been dealing so far with wave functions. Our wave functions, where these functions of x and t, they represent the dynamics of your system, the dynamics of your particle as it moves in time. But time, as you are seeing in quantum mechanics, is a little bit of a spectator. It's an arena where things happen. But the operators, and most of the interesting things, are going on without reference to time.

Time evolution, you have an expansion of a wave function in terms of energy, eigenstates, at a given time. And then you can evolve it easily with the way we've learned, adding e to the minus i et over h bar for each energy eigenstate. So time will play no role here. So when I talk about the wave function, at this moment you could put the time, but we will talk about the wave functions that have no time dependence.

So, say, a psi of x wave function. So this psi of x may be the true wave function at time equals 0, or you could just simply think of it as the psi of x. Now, this wave function means that we're treating x in a particular way, and we say that we're working in the x representation, the position representation.

Now, this means that we have an easy way to figure out what this operator does when it acts on this function. So what it acts on this function, it will give you another function, and the definition of this is that the position operator acting on the function psi of x is defined to be another function, which is the function x times psi of x.

Well, we're talking about these wave functions and operators on wave functions. And a recurrent theme in quantum mechanics is that we will think of wave functions, sometimes we call them states. Sometimes we call them vectors. And we basically think of wave functions as vectors. And things that act on wave functions are the things that act on vectors. And the things that act on vectors, as you know in mathematics, is matrices. So we're compelled, even at this early stage, to get a picture of how that language would go if we're talking about these things.

So how do we think of a wave function as a vector? And how do we think of x as a matrix? So there's a way to do that. It will not be totally precise, but it's clear enough. So suppose you have a wave function, and we're interested in its values from 0 up to a. This wave function is a function of x between 0 and a. So it's the psi of x for x between a and 0. That's all the information.

What we're going to do is we're going to divide this thing, this line, this segment, into a lot of pieces. And we're going to say, look, instead of writing a function like sine of x or cosine of x, let's just give the values and organize them as if this will be a vector of many components. So let's divide this in sizes epsilon, such that N times epsilon is equal to a. So there are N of these intervals.

So we think of psi as a vector whose first component is psi at 0. The second is psi at epsilon. The third is psi at 2 epsilon. And the last one is psi at N epsilon. And depending on how much accuracy you want to work with, you take epsilon smaller and larger, keeping a constant. And this would be like summarizing all the information of a function in a vector.

Now, that's intuitively a nice way to think of it. May look, with your background in classical physics, a little strange that we sort of put the value at 0 along the x-axis, first component, the value at epsilon along the y, the value of 2 epsilon along the z. But we need more axes. So you need many axes here. In this case, this is a N plus 1 column vector. It has N plus 1 entries, because 0 up to N, that's N plus 1 entries.

But that's a fine way of thinking of it. Not exact because we have an epsilon. In this way of thinking about the wave function, we can then ask, what does the matrix x hat look like? So x hat is an operator, and it acts this way. So here is how it looks like. We would think of x hat as an N plus 1 times N plus 1 matrix. And its entries are 0 everywhere, except in the diagonal, where they are 0 epsilon, 2 epsilon, up to N epsilon. And here is a big 0 and a big 0.

This, I claim, is the way you should think of the x operator if you thought of the wave function the way we wrote it. And how do we check that? Well, x operator acting on psi should be this acting on that. And then, indeed, we see that if x hat is acting on psi of x, what do we get?

Well, it's easy to multiply a diagonal matrix times a vector. Here you get 0 times psi of 0. You get a vector, so let me make this thinner. Then I get epsilon times psi of epsilon, 2 epsilon times psi of 2 epsilon, up to N epsilon times psi of N epsilon.

And indeed, that matrix looks like the matrix associated with this wave function because here is the value at 0 of this wave function. Here is the value at epsilon of this wave function, and so on. So this has worked out all right. We can think of the wave function as a column vector, and then the position operator as this vector as well.

Now, given that we know how the x operator is defined, we can also think easily about what is the expectation value of x on a wave function. Something that you really know, but now maybe becomes a little clearer. Here you're supposed to do psi star of x times the x operator acting on psi of x. But we have the definition of this, so this is, as you imagine, dx-- and I should put primes maybe, well, I don't have to put primes-- dx psi star of x x psi of x, which is what you would have done anyway.

Well, given that we've started with this, we can ask also, is there eigenstates of the x operator? Yes, there are. but then fortunately, are a bit singular. So what should be an eigenstate of x? It's some sort of state. Intuitively, it has a definite value of the position. So it just exists for some value of x. So it's naturally thought as a delta function.

So let me define a function, psi sub x0 of x. So it's a function of x labeled by x0, and define it to be delta of x minus x0. So I claim that is an eigenstate of x hat. x hat on psi x0 of x is equal, by definition, to x times psi x0 of x, which is x times delta of x minus x0.

And when you multiply a function of x times a delta function in x, it is possible to evaluate the function that is being multiplied by the delta function at the place where the delta function fires. It has the same effect on integrals or anything that you would do. So here, this is equal to x0 times delta x minus x0. You evaluate the x at x0.

And finally, this is x0 times that function psi x0 of x. And therefore, you've shown that this operator acting on this function reproduces the function-- that's the definition of eigenstate as an operator-- and the eigenvalue is the number that appears here, and it's x0. So this function is an eigenstate of x hat with eigenvalue, e.v., x0.

The only complication with this eigenfunction is that it's not normalizable. So it doesn't represent the particle. It can be used to represent the particle, but it's a useful function. You can think of it as something that can help you do physics, and don't insist that it represents a particle.

So this is the story for position. And the position gets actually more interesting as soon as you introduce the dual quantity, momentum. So what is momentum here? So momentum is an operator, and this operator must be defined.

Now, you had a shorthand for it in 804, which is p hat equal h bar over i d dx. And this shorthand means actually that, in what we call the position representation where we're using wave functions that depend on x, well, the momentum is given by this operator.

And the story of why this was the case was sort of something that was elaborated on in 804, the work of de Broglie, that saw that the wavelength of the wave has to do with the momentum of a wave. And finally, people understood that this would measure the momentum of the wave.

So this is the operator. And therefore, in the representation that we're working-- representation is a word that has a lot of precise meaning, but now I'm just using it in the sense that, well, we're working either with x's or with p's. And we're working with x's. That's why p looks like something to do with x.

So what is p hat on a wave function? Well, that's what this means. It's another wave function obtained by taking the x derivative. So that's the definition of it acting on a wave function.

The one thing that must be verified, of course, is that this definition is consistent or implies this commutation relation. So you've defined it as an operator. x, we've defined it as an operator. But most of us think that it doesn't look like an operator is multiplying. But it is an operator.

So this one does look like an operator. It's a differential operator. And you can try to see if this equation is true. And the way to test these commutators is something that, again, I don't think is unfamiliar to you, but let's go through it, is that you try to evaluate this product of operators acting on a wave function.

And if things work out well, we'll see you should get ih bar times that wave function. If that is the case, you say, OK, I've proven that equation, because it's an operator equation. The left-hand side of that equation is the product in different orders of two operators, therefore it's an operator. The right-hand side is another operator. It's the operator multiplied by ih, anything that you'll get.

Well, if this is an operator identity, the operator on the left must be equal to the operator on the right, which just means that, acting on anything, they must give the same answer. So if I managed to prove that this is equal to this, I've proven that for anything that is the answer. And therefore, I can write the top one.

And let me just do it, even though this may be kind of familiar to many of you. It's good to do this slowly once in your life. So let's go through this. So this says x operator p operator on psi minus p operator of x operator on psi. When you have several operators, like ABC acting on psi, this really means let C act on psi, and then let B act on C psi, and then let A act on that. The operators act one by one. The closest one acts first.

So here I'm supposed to let B act on psi, but that means that thing. So now x is acting on h over i d psi dx. On this one, I have p acting on x psi, because that's what x hat psi is. Here, this is multiplication by x of a function of x. So this is just h over i x d psi dx. And here, I have h over i d dx of this whole thing x psi.

And you can see that when you act here, you act first on the x, and you get something. And then you act on the psi, and you get this same term. So the only contribution here is equal to minus h over i, the d dx on x times psi, which is ih bar psi, which is what I wanted to show. So this is true. And therefore, you could say that this definition is consistent with your definition of x, and they represent this operator.

One more thing you could try to do, and it's fun to do it, is we had a matrix representation for x. Can I think of p as a matrix? How would you do it? What kind of matrix would p look like? Well, yes?

AUDIENCE: You just generate a finite difference equation.

PROFESSOR: You could do it, exactly, with taking finite differences. So for example, if you think that you want to produce the wave function psi prime at 0, psi prime at epsilon, psi prime, that's what the derivative gives you, you'll write this as 1 over epsilon, say, psi at epsilon minus psi at 0. That's the derivative at 0 roughly. It would be psi at 2 epsilon minus psi at 0 over 2. And you could build it. You could build it. I'm not going to do it. You may want to do it and try and see how the derivative operator looks as a matrix.

And then if you really want to spend some time thinking about it, you could try to see if this matrix and this matrix commute to give the right answer. And as you try it, you will figure out all kinds of funny things that we will talk about later in the course. So you can represent the momentum operator as a matrix indeed, and there are interesting things to say about it, and it's a good subject.

So let's continue with the momentum and ask for eigenstates of the momentum. So eigenstates of p, you know them. They're e to the ipx things. So let's write them with some convenient normalization. This is an [INAUDIBLE] wave function that depends on x with momentum p. And we'll write it, as a definition, as e to the ipx over h bar, and I'll put it a 2 pi h bar here. It's kind of a useful normalization.

Then p hat on psi p of x, well, p hat is supposed to take h over i d dx, and take h over i d dx of psi p. And h over i cancels the i over h. When you take the d dx, you get p out, and you get the same wave function. So indeed, you get p times psi p of x. So indeed, this is the eigenstate of the momentum operator, and it has momentum p.

Well, what is the use of this? Well, say you have a representation, what we call the position representation, of the wave function and operators. Let us think now of the momentum representation. So what does all that mean?

Well, there is the Fourier transform operation in which we have psi of p. Well, let me write it this way, actually. I'll write any psi of x physically can be represented as a sum of momentum eigenstates. Therefore, that's Fourier's theorem, minus infinity to infinity dp e to the ipx over h bar square root of 2 pi h psi tilde of p. That's Fourier transformation, defines psi tilde of p.

And Fourier's theorem is the fact that not only you can do that, but you can invert it so that psi tilde of p can also be written as an integral, this time over x from minus infinity to infinity e to the minus ipx over h bar, also 2 pi h bar psi of x.

So let's ponder this equation for a couple of minutes. Well, as a physicist, you think of this, well, this is telling me that any wave function could be written as a superposition of momentum eigenstates. Here are the momentum eigenstates.

And for each value of momentum, you have some coefficient here that tells me how much of that momentum eigenstate I have. Now, here is the opposite one. Psi tilde of p and psi of x are related in this way. So these coefficients, if you want to calculate them, you calculate them this way.

But now let's think of it as a change of representation. The physics is contained in psi of x. All what you wish to know about this physical system in quantum mechanics is there in psi of x. But it's also there in psi of p, because they contain the same information. So there are different ways of encoding the same information.

What is the relation between them? This, we thought of it as a vector, vector in position space, an infinite dimensional space that is talking about positions. This is another vector in momentum space. Think of it now the infinite line. So this is an infinite vector with all those points little by little, from minus infinity to plus infinity, all of them there, gigantic vector. And here is another gigantic vector with p from minus infinity to infinity. And in between, there's an integral.

But now, with your picture of quantum mechanics, you see an integral, but you also see a matrix. And what is this matrix? Think of this as some sort of psi sub p. And this as some sort of matrix, px psi x, in which if you have a product-- you'll remember when you multiply matrices, a matrix on a vector, you sum over the second index. That's the product for matrix. And then the first index is the index here.

So here is what it, more or less, is like. Psi tilde of p [? subtend ?] by this, and this matrix depends on two labels, p and x, and it's that. So it's a matrix full of phases. So how do you pass from the coordinate representation of the information, a vector of all values of the wave function in all positions? By multiplying with this matrix of phases that is here, and it gives you this representation.

So different representations means using different vectors to represent the physics. And this vector is a very nice one. And because of these properties of the momentum operator and all these things, this vector is also a very nice one. And there's an integral transform or some sort of infinite matrix product that relates them.

And we shouldn't be uncomfortable about it. That's all fine. So we say that we have, for example, psi of x as one representation of the state and psi tilde of p as another representation of the same physics.

We can do one more thing here, If I continue. We can take that boxed equation on the blackboard up there and act with h bar over i d dx on psi of x. So that is equal to h i d dx, and I'll write what psi of x is, is minus infinity to infinity dp e to the ipx over h bar square root of 2 pi h bar psi tilde of p.

Now, when we act on this, as you know, h bar over i d dx just acts on this and produces the factor of p. So this is equal to minus infinity to infinity dp e to the ipx over h bar over square root of 2 pi h bar p times psi tilde of p.

So look at this equation again. This double arrow is to mean that there are equivalent physics in them. They have the same information. It's the same data encoded in a different way. And that different way, this arrow is Fourier transformation. And this Fourier transformation is explained here.

So now you have Fourier transformation the same way. So here we have-- what we've learned is that h over i d dx of psi is represented in momentum space by p psi tilde of p. And this was p hat acting on psi of x. So the corresponding thing in momentum space of p hat acting on psi of x is p multiplying psi tilde of p, which is to say that we can think of the abstract operator p hat acting on psi tilde of p as just p psi tilde of p.

So in momentum space, the operator p hat acts in a very easy way. In coordinate space, it takes derivatives. In momentum space, it's multiplicative. So in position space, x is multiplicative. But in momentum space, x would not be multiplicative. x would also be a derivative. So I leave it for you as an exercise to show that or convince yourself in several ways, that x hat is really i h bar d dp in p space, in i h bar d dp.

All right. So that's really all I wanted to say about position and momentum operators at this moment. They will come back when we'll introduce bra-ket notation in detail. We'll revisit this a little. But the main concepts have been illustrated. Are there questions? We're about to leave this, so if you have any questions at this moment. Yes?

AUDIENCE: Could you explain again how you used this [INAUDIBLE] h bar over i d dx assign to [INAUDIBLE]?

PROFESSOR: Right. So the question was, why did I associate these things? So it really goes back here to what the meaning of this arrow is. The meaning of this arrow is Fourier transformation. So this psi tilde and psi of x are related in this way. That's Fourier transformation, and that's what we mean by this arrow. It also means that whatever physics you have here, you have it there.

So really, when you have something acting on a state, for example, if you have some operator acting in here, well, you get a new wave function. And there should be one on the right that corresponds to it, that has the same information as the one in which you've acted with something.

So what we claim here is that, also in the sense of Fourier transformation or having the same information, h bar over i, the derivative of psi, is encoded by this. So we say, thinking abstractly, what is this? This is the momentum operator. Therefore, I'm going to say that the momentum operator really is the same momentum operator, whether it acts on wave functions that you show them to mean this way or wave functions that, because you're in another mood, you decide to give them to me in momentum space.

So as you change your mood, the operator takes different forms but is doing the same thing. It's totally reversible. It's acting on that-- you see, the operator is always the same, but you give me the data in two different ways, then the operator has to do the thing in a different way. So that's what it means that the operator has different representations. In the [INAUDIBLE] representation, it looks like a derivative. In the momentum representation, it looks like multiplying. Other questions? Yes?

AUDIENCE: So by saying that they sort of represent [INAUDIBLE] to the same positions, does that mean that h bar over i p e to the xi and p psi p are like the same [INAUDIBLE]?

PROFESSOR: That h bar over d dx psi and p-- yeah. They are the same data, the same state represented in different ways. Yeah.

All right. So time for a change. We're going to talk about Stern-Gerlach and spin. Now, spin will keep us busy the biggest chunk of this semester. So it will be spin-1/2, and we're really going to go into enormous detail on it. So this is just the beginning of the story that will be elaborated at various stages.

So at this moment, I will talk about this experiment that led to the discovery of spin, and if you try to invent the theory that describes this experiment, what you would possibly begin doing. And then we go through the mathematics, as I mentioned to you, for maybe a week and a half or two weeks, and then return to the spin with more tools to understand it well.

So the subject is the Stern-Gerlach experiment, Stern-Gerlach experiment. So the Stern-Gerlach experiment was done in Frankfurt, 1922. It was an experiment that, in fact, people were extraordinarily confused. It was not clear why they were doing it. And for quite a while, people didn't understand what they were getting, what was happening with it.

In fact, Pauli had thought that the electron has like two degrees of freedom and didn't know what it was, those two degrees of freedom. Kronig suggested that it had to do somehow with the rotation of the electron.

Now, Pauli said that's nonsense. How can an electron rotate and have angular momentum because it has a rotation? It would have to rotate so fast, even faster than the speed of light to have the angular momentum, and then this little ball that would be the electron would disintegrate. And it made no sense to him that there would be such a thing. So Kronig didn't publish this.

Then there were another two people, Uhlenbeck and Goudsmit, at the same time, around 1925, had the same idea, angular momentum of this particle. And their advisor was Ehrenfest, and said it doesn't make too much sense, but you should publish it.

[LAUGHTER]

And thanks to their publishing, they are given credit for discovering the spin of the electron. And Pauli, a couple of years later, decided, after all, I was wrong. Yes, it is spin, and it's all working out. And 1927, five years after the experiment basically, people understood what was going on.

So what were these people trying to do? First, Stern and Gerlach were atomic physicists, and they were just interested in measuring speeds of thermal motion of ions. So they would send beams of these ions and put magnetic fields and deflect them and measure their velocities. And eventually, they were experts doing this kind of thing.

And they heard of Bohr, that said that the electron has angular momentum and is going around the proton in circles, so it might have angular momentum. They said, oh, if it has angular momentum because it's going around the proton, maybe we can detect it. And when they did the experiment, they got something. And they said, well, we're seeing it.

But it was not that. They were not seeing the orbital angular momentum of the electron because that electron in these silver atoms actually has no angular momentum, as we will see, no orbital angular momentum. It only has spin. So they were actually seeing the spin. So it was a big confusion. It took some time.

Basically, they took the beam, and they split it with a magnetic field, and the clean split was something nobody understood. So they called it space quantization, as of it's separated in space. Space is quantized. A pretty awful name, of course. There's nothing quantized about space here. But it reflects that when you don't know what's really happening, your names don't come out too well.

So what we have to understand here, our goal today is to just see what's happening in that experiment, quantify a bit the results, and then extract the quantum mechanical lessons from it. So let us begin with the important thing. You don't see the spin directly. What you see is magnetic moments. So what's that? So what are magnetic moments? Magnetic moments, mu, is the analog, the magnetic analog of an electric dipole. A mu is called a magnetic dipole. You say it has a magnetic moment.

And the magnetic moment is given by I times the area. What does that mean? Well, a precise discussion would take some time. But roughly, you can simplify when you think of a loop that is in a plane, in which case there's an area associated to it. And if the loop is this one, the area is defined as the normal vector to the oriented loop.

So an oriented loop has an area vector. And the orientation could be focused the direction of the current. There is some area. And the magnetic moment is given by this thing. It points up in the circumstances when this current goes like that.

So that's a magnetic moment. A little bit of units. The way units work out is that mu B-- magnetic moments and magnetic fields have units of energy. So magnetic moments you could define as energy, which is joules, divided by tesla, or ergs divided by gauss, because mu B has units of energy.

So how do magnetic moments originate in a charge configuration? Well, you can simply have a little current like that. But let's consider a different situation in which you have a ring of charge, a ring of charge of some radius R. It has a total charge Q, and it has a linear charge density lambda. It's uniform, and it's rotating with some velocity v. If you wish, it also has a mass M. There are all kinds of [? parameters. ?] How many? Mass, charge, radius, and velocity.

Here we go. We have our solid ring of charge rotating, and we want to figure out something quite fundamental, which is the origin of this principle. We said, you really never see spins directly. You never see this intrinsic angular momentum directly. You see magnetic moments.

But then actually what happens is that there's a universal relation between magnetic moments and angular momentum. This is a key concept in physics. Maybe you've seen it before. Maybe you haven't. Probably you might have seen that in 802.

So how does that go? Let's calculate the magnetic moment. So the current is the linear charge density times the velocity. The linear charge density is the total charge divided by 2 pi R times the velocity.

Now the area, to give the magnetic moment, we'll have mu is equal to I times the area. So it would be this Q times 2 pi R v times the area, which would be pi R squared. So the pi's cancel, and we get 1/2 QvR. OK. 1/2 QvR, and that's fine and interesting. But OK, depends on the radius, depends on the velocity.

So here is the magnetic moment is supposed to be going up. But what else is going up? The angular momentum of this thing is also going up. So what is the magnitude of the angular momentum L? L is angular momentum. Well, it's the mass times the momentum-- it's the mass momentum cross R, so MvR. The momentum of R cross p for each piece, contributes the same, so you just take the total momentum. This really is 0, but add them up little by little, and you've got your MvR.

So here you have vR, so here you put 1/2 Q over M MvR. And you discover that mu is equal to 1/2 Q over M L. So maybe write it better-- Q over 2M L. I'm sorry, this is the normal. The M shouldn't change, M.

And I box this relation because an interesting thing has happened. All kinds of incidentals have dropped out. Like the velocity has dropped out. The radius has dropped out as well. So if I have one ring with this radius and another ring with a bigger radius, the relation between mu and L is the same, as long as it's rotating with the same speed.

So this is actually a universal relation. It is not just true for a little ring. It's true for a solid sphere or any solid object axially symmetric. It would be true. You could consider any object that is axially symmetric, and then you start considering all the little rings that can be built. And for every ring, mu over L is the same, and they all point in the same direction. Therefore, it's true under very general grounds. And that is a very famous relation.

So now you could speculate that, indeed, the reason that a particle may have a magnetic moment if it's made by a little ball of charge that is rotating. But that was exactly what Pauli didn't like, of course. And you would like to see what's really happening with particles.

So when you think of a true quantum mechanical particle, let's think of a particle in general, a solid particle rotating. We'll change the name to S for spin angular momentum. Because that little part, this is just one particle. We're not thinking of that little particle going around a nucleus. We're thinking of that little particle rotating. So this is a little piece of that little particle that is rotating.

So you could ask, if, for the electron, for example, is it true that mu is equal to e over 2 mass of the electron times its spin? So this would be a vindication of this classical analysis. It might be that it's related in this way. So actually, it's not quite true, but let's still improve this a little bit.

In terms of units, we like to put an h bar here and a 2Me. And put spin here, angular momentum, divided by h. Because this has no units, h bar has the units of angular momentum, x times p. It's the same units, so units of angular momentum. So h bar would be convenient. So that over here, you would have units of a dipole moment, or magnetic moment, magnetic moment units.

So what does happen for the electron? Well, it's almost true, but not quite. In fact, what you get is that you need a fudge factor. The fudge factor is that, actually, for elementary particles, you have a g, which is a constant, which is the fudge factor, e h bar 2 over M of the particle S over h bar. Sometimes called the Lande factor.

You must put a number there. Now, the good thing is that the number sometimes can be calculated and predicted. So when people did this, they figured out that for the electron the number is actually a 2. So for the electron, g of the electron is equal to 2.

Now that, you would say, cannot be an accident. It's twice what you would predict sort of classically. And the Dirac equation, the relativistic equation of the electron that you have not studied yet but you will study soon, predicts this g equal to 2. It was considered a great success that that equation gave the right answer, that people understood that this number was going to be 2.

So for the electron, this is 2. So this quantity is called-- it's a magnetic dipole moment-- is called mu B for Bohr magneton. So how big is a mu B? It's about 9.3 times 10 to the minus 24 joules per tesla.

AUDIENCE: Professor.

PROFESSOR: Yes?

AUDIENCE: [INAUDIBLE]. So where exactly does the fudge factor come in? Is it just merely because [INAUDIBLE]?

PROFESSOR: Right. So the classical analysis is not valid. So it's pretty invalid, in fact. You see, the picture of an electron, as of today, is that it's a point particle. And a point particle literally means no size. The electron is not a little ball of charge. Otherwise, it would have parts.

So an electron is a point particle. Therefore, a point particle cannot be rotating and have a spin. So how does the electron manage to have spin? That you can't answer in physics. It just has it. Just like a point particle that has no size can have mass. How do you have mass if you have no size? You get accustomed to the idea. The mathematics says it's possible. You don't run into trouble.

So this particle has no size, but it has an angular spin, angular momentum, as if it would be rotating. But it's definitely not the case that it's rotating. And therefore, this 2 confirms that it was a pointless idea to believe that it would be true. Nevertheless, kind of unit analyses or maybe some truth to the fact that quantum mechanics changes classical mechanics. Turns out that it's closely related.

For the proton, for example, the magnetic moment of the proton is quite complicated as well because the proton is made out of quarks that are rotating inside. And how do you get the spin of the proton and the magnetic moment of the proton? It's complicated.

The neutron, that has no charge, has a magnetic moment, because somehow the quarks inside arrange in a way that their angular momentum doesn't quite cancel. So for example, the value for a neutron, I believe, is minus 2.78 or something like that. It's a strange number.

Another thing that is sort of interesting that is also true is that this mass is the mass of a particle. So if you're talking about the magnetic moment of the proton or the neutron, it's suppressed with respect to the one of the electron. The electron one is much bigger because, actually, the mass shows up here. So for a neutron or a proton, the magnetic moment is much, much smaller.

So, in fact, for an electron then, you would have the following. Mu is equal to minus g, which is 2, mu B S bar over h. And actually, we put the minus sign because the electron has negative charge. So the magnetic moment actually points opposite.

If you rotate this way, the angular momentum is always up. But if you rotate this way and you're negative, it's as if the current goes in the other direction. So this is due to the fact that the electron is negatively charged. And that's the final expression. So OK, so that's the general story with magnetic moments.

So the next thing is, how do magnetic moments react when you have magnetic fields? So that is something that you can calculate, or you can decide if you have a picture. For example, if you have a loop of charge like this, and you have magnetic field lines that go like this, they diverge a bit.

Let me see you use your right-hand rule and tell me whether that loop of current will feel a force up or down. I'll give you 30 seconds, and I take a vote. Let's see how we're doing with that. And I'll prepare these blackboards in the meantime. All right. Who votes up? Nobody. Who votes down? Yeah, [INAUDIBLE]. Down, exactly.

How do you see down? Well, one way to see this, look at the cross-section. You would have this wire here like that. The current is coming in on this side and going out this way. Here you have the field lines that go through those two edges, and the magnetic field is like that.

And the force goes like I cross B. So I goes in, B goes out. The force must be like that, a little bit of force. In this one, I cross B would be like that, a little bit of force. Yep. Has a component down because the field lines are diverging.

So what is the force really given by? The force is given by the gradient of mu dot B. This is derived in E&M. I will not derive it here. This is not really the point of this course. But you can see that it's consistent. This is saying that the force goes in the direction that makes mu dot B grow the fastest. Now mu, in this case, is up. So mu dot B is positive, because mu and the magnetic field go in the same direction. So mu dot b is positive.

So the force will be towards the direction-- that's what the gradient is-- that this becomes bigger. So it becomes bigger here, because as the field lines come together, that means stronger magnetic field. And therefore, mu dot B would be larger, so it's pointing down. If you have a magnetic field that is roughly in the z direction, there will be a simplification, as we will see very soon.

So what did Stern and Gerlach do? Well, they were working with silver atoms. And silver atoms have 47 electrons, out of which 46 fill up the levels and equal 1, 2, 3, and 4. Just one lone electron, a 5s electron, the 47th electron, it's a lonely electron that is out in a spherical shell, we know now with zero orbital angular momentum. It's an S state.

And therefore, throwing silver atoms through your apparatus was pretty much the same thing as throwing electrons, because all these other electrons are tied up with each other. We know now one has spin up, one spin down. Nothing contributes, no angular momentum as a whole. And then you have this last electron unpaired. It has a spin. So it's like throwing spins.

So moreover, throwing spins, as far as we're concerned, Stern and Gerlach wouldn't care. Because of these relations, it's throwing in dipole moments. And they would care about that because magnetic fields push dipole moments up or down.

Therefore, what is the apparatus these people had? It was sort of like this, with an oven, and you produce some silver atoms that come out as a gas, a collimating slit. Then you put axes here-- we put axes just to know the components we're talking about. And then there's magnets, some sort of magnet like this, and the screen over there. So basically, this form of this magnet that I've tried to draw there, although it's not so easy, if I would take a cross-section it would look like this.

So the magnetic field has a gradient. The lines bend a bit, so there's a gradient of the magnetic field. And it's mostly in the z direction, so z direction being pointed out here. So there's the magnetic field. The beam then comes here. And the question is, what do you get on this screen? Now, I have it a little too low. The beam comes there and goes through there.

So the analysis that we would have to do is basically an analysis of the forces. And relatively, we don't care too much. The fact is that there's basically, because the magnetic field is mostly in the z direction and varies in z direction, there will be a force basically in the z direction.

Why is that? Because you take this, and you say, well, that's roughly mu z Bz, because it's mostly a magnetic field in the z direction. And mu is a constant, so it's basically gradient of Bz. Now, that's a vector. But we're saying also most of the gradient of Bz is in the z direction, so it's basically dBz dz.

Now, there is some bending of the lines, so there's a little bit of gradient in other directions. But people have gone through the analysis, and they don't matter for any calculation that you do. They actually average out. So roughly, this gradient is in the z direction. I'm sorry, the gradient is supposed to be a vector.

So you get a force in the z direction. And therefore, the thing that people expected was the following. You know, here comes one atom, and it has its magnetic moment. Well, they've all been boiling in this oven for a while. They're very disordered. Some have a z component of magnetic-- the magnetic moment is pointing like that, so they have some component, some down. Some are here. They have no component.

It's all Boltzmann distributed all over the directions. Therefore, you're going to get a smudge like this. Some ones are going to be deflected a lot because they have lots of z component of angular momentum or z magnetic moment. Others are going to be deflected little.

So this was the classical expectation. And the shock was that you got, actually, one peak here, an empty space, and another peak there. That was called space quantization. Stern and Gerlach worked with a magnetic field that was of about 0.1 tesla, a tenth of a tesla. And in their experiment, the space quantization, this difference, was 1/5 of a millimeter. So not that big, but it was a clear thing. It was there.

So everybody was confused. They thought it was the orbital spin, angular momentum that somehow had been measured. At the end of the day, that was wrong. It couldn't have been that. People understood the Bohr atom, realized, no, there's no angular momentum there.

The idea of the spin came back, and you would have to do a calculation to determine what is the value of the spin. So the exact factor took a while to get it right. But with the idea that mu z is equal to minus 2 Bohr magenton Sz over h bar, which we wrote before.

Well, mu z, if you know the strength of your magnetic field, you can calculate the deflections. You know what mu B is. So therefore, you get the value for Sz over h. And experiments suggested that Sz over h was either plus or minus 1/2. And this kind of particle, it has Sz over h bar equal plus or minus 1/2, is called the spin-1/2 particle. So again, from this equation, this can be measured. And you then use this, and you get this value.

So the experiment is a little confusing. Why did this happen? And how do we think of it quantum mechanically? Now 804 sort of began with these kind of things. And you know by now that what's happening is the following, that somehow, mathematically, every state is a superposition of a spin up and a spin down. So every particle that goes there has half of its brain in the spin up and half of its brain in the spin down.

And then as it goes through the magnetic field, this thing splits, but each particle is in both beams still. And they just have this dual existence until there's a screen and there's detectors. So they have to decide what happens, and then either collapses in the top beam or lower beam. Nothing happens until you put the screen.

That's what we think now is the interpretation of this experiment. But let's use the last few minutes to just write this in terms of boxes and get the right ideas. So instead of drawing all that stuff, we'll draw a little box called a z hat box, a Stern-Gerlach apparatus.

In comes a beam, out would come two beams, Sz equal h bar over 2 and Sz equal minus h bar over 2. And the convention is that the plus goes up and the minus goes down, which I think is probably consistent with that drawing. And that's the Stern-Gerlach apparatus. It measures Sz, and it splits the beam. Each particle goes into both beams until there's a device that measures and decides where you go.

So you can do the following arrangements. So here's arrangement number 1, a Stern-Gerlach device with z. Then you block the lower one and let the top one go as Sz equal h bar over 2. And then you put another Stern-Gerlach machine, z hat, that has two outputs. And then you ask, what's going to happen? And the experiment can be done and, actually, there's nothing here coming out, and all the particles come out here with Sz equal h bar over 2.

What are we going to learn from this? In our picture of quantum mechanics, we're going to think of this as there are states of the electron that have-- and I will write them with respect to z-- they have plus h bar over 2 and states that have minus h bar over 2. And what we will think is that these are really old basis states, that any other state, even one that points along x, is a superposition of those two.

This is a very incredible physical assumption. It's saying this system is a 2-dimensional complex vector space, two vectors, two unit, two basis vectors. And from those two, all linear combinations that are infinite represent all possible spin configurations.

And what is this saying? Well, as we will translate it into algebra, we will say that, look, here is a state plus. And when you try to measure, if it had any minus component, it had nothing. So we will state that as saying that these states are orthogonal. The minus state and the plus state have zero overlap. They are orthogonal basis states.

And, for example, well, you could also do it this way. That would also be 0. And you could also say that z plus and z plus is 1, because every state that came in as a plus came out as a plus. They had perfect overlap. So these are two orthonormal basis vectors. That's what this seems to suggest.

And it's a little strange, if you think, because there's a clash between arrows and the notion of orthonormality. In 3-dimensional vectors, you think of this vector being orthogonal to this. But you wouldn't think of this vector as being orthogonal to that one.

And here is the spin is up, and this is the spin down. And those two are orthogonal. You say, no, they're anti-parallel. They're not orthogonal. No, they are orthogonal. And that's the endlessly confusing thing about spin-1/2.

So these states, their pictures of the spins are arrows. But don't think that those arrows and the dot product give you the orthogonality, because this is up and down. If you would be doing the dot product of an up and down vector, you would not get 0. But this is 0.

Then you do the following experiment. So let's do the next one. And the next one is, again, the z filter. Take this one, block it. Then you put an x filter. And what actually happens is that you would get states with Sx, now, h bar over 2 and Sx equal minus h bar over 2, because it's an x filter. The magnetic field is a line in the x direction.

Now, all these things have Sz equal h bar over 2. And what happens in the experiment is that 50% of the particles come out here and 50% come out there. So a spin state along the x direction has some overlap with a spin state along the z direction. Normal vectors, a z vector and an x vector, are orthogonal.

Not here for spins. The spin pointing in the z and the spin pointing in the x are not orthogonal states. They have overlaps. So this means that, for example, the x plus state and the z plus state have an overlap. This is notations that-- we're going to be precise later. But the same thing with the x minus state, it has an overlap, and somehow they're about the same.

Finally, the last experiment is this, z hat, block again, x hat, but this time block one. So here is a state with Sx equals minus h bar over 2. Here is a state with Sz equal h bar over 2. And now you put the z machine again. And what happens?

Well, there's two options. People who were inventing quantum mechanics no wonder thought about them. Here they could say, look, I filtered this thing, and now all these electrons have Sz equal h bar over 2. And now all these electrons have Sx equal minus h bar over 2. Maybe, actually, all these electrons have both Sz equal h over 2 and that because I filtered it twice. So it maybe satisfies both.

So if all these electrons would have Sz equals h over 2 and this, then you would only get something from the top one. But no, that's not what happens. You get in both. So somehow, the memory of these states coming from Sz equals h over 2 has been destroyed by the time it turned into a state with Sx equal minus h over 2. And a state cannot have simultaneously this and that. That's two properties, because you get 50% here and 50% there.

So we'll discuss next time a little more about these relations and how can the states be related, the ones that we use as the basis vectors and all the others along x and others that we could build some other way.

All right. See you next week. There's office hours today, 5:00 to 6:00, Monday, 4:30 to 5:30.