PROFESSOR: Today we have plenty to do. We really begin in all generality the addition of angular momentum. But we will do it in the set up of a physical problem.

The problem of computing the spin orbit interactions of electrons with the nucleus. So this is a rather interesting and complicated interaction. So we'll spend a little time telling you about the physics of this interaction.

And then once the physics is clear, it will become more obvious why we have to do these mathematical contortions of adding angular momentum in order to solve this physical problem. So it's a sophisticated problem that requires several steps.

The first step is something that is a result in perturbation theory. Feynman Hellman Theorem of perturbation theory. And that's where we begin.

So it's called Feynman Hellman Theorem. It's a very simple result. Theorem.

And we'll need it in order to understand how a small perturbation to the Hamiltonian changes the energy spectrum. So we have H of lambda be a Hamiltonian with a parameter in lambda. Lambda.

And psi n of lambda being normalized energy eigenstate with energy, En of lambda. So that's the whole assumption of the theorem. We have a Hamiltonian. It depends on some parameter that we're going to vary.

And suppose we consider now an eigenstate of this Hamiltonian that depends on lambda, so the eigenstate also depends on lambda. And it has an energy, En of lambda. So the purpose of this theorem is to relate these various quantities.

And the claim is that the rate of change of the energy with respect to lambda can be

computed pretty much by evaluating the rate of change of the Hamiltonian on the relevant states. So that's the claim. And it's a pretty nice result.

It's useful in many circumstances. And for us will be a way to discuss a little perturbation theory. Perturbation theory is the subject of 806 in all details. And it's a very sophisticated subject. Even today we were going to be finding that it's not all that easy to carry it out.

So how does this begin? Well, proof. You begin by saying that En of lambda is the energy eigenstate, is nothing else but psi n of lambda. H of lambda. Psi n of lambda.

And the reason is, of course, that H and psi n is En of lambda times psi n of lambda. And this is a number goes out. And the inner product of this things is 1, because the state is normalized.

So this is a good starting point. And the funny thing that you see already is that, in some sense, you just get the middle term when you take the derivative with respect to lambda. You don't get anything from these two.

And it's simple in fact. So let me just do it. V, En, V lambda would be the term that Feynman and Hellman gave.

V, H, V lambda, psi n of lambda. Plus one term in which we differentiate this one. V, d lambda of the state psi n of lambda.

Times H of lambda, psi n of lambda. Plus the other term in which you differentiate the ket. So psi n of lambda, H of lambda, d, d lambda of psi n of lambda.

And the reason these terms are going to vanish is that you can now act with H again on psi n. H is supposed to be Hermitian, so it can act on the left. And therefore, these two terms give you En of lambda, times d d lambda of psi n of lambda-- psi n of lambda-- plus the other term, which would be psi n of lambda, the bra times the derivative of the ket.

But this is nothing else than the derivative of the inner product. In the inner product

to differentiate-- the inner product differentiates the bra, it differentiates the ket. And do it.

And this thing is equal to 1, because it's normalized. So this is 0. End of proof. These two terms vanish. And the result holds.

Yes?

AUDIENCE: How do you know it stays normalized when you vary lambda?

PROFESSOR: It's an assumption. The state is normalized for all values of n. So if you have a state that you've constructed, that is normalized, you can have this result. So it's an assumption. You have to keep the state normalized.

Now this is a baby version of perturbation theory. It's a result I think that Feynman did as an undergrad. And as you can see, it's very simple.

Calling it a theorem is a little too much. But still, the fact is that it's useful. And so we'll just go ahead and use it.

Now I want to rewrite it in another way. So, suppose you have a Hamiltonian, H, which has a term H0, plus lambda, H1. So, the parameter lambda, H of lambda, is given in this way.

And that's a reasonable H of lambda. Sometimes, this could be written as H0 plus something that we will call the change in the Hamiltonian. And we usually think of it as a small thing.

So what do we have from this theorem? From this here we would have the d, En, d lambda is equal to psi n of lambda, H1, psi n of lambda. Now, we can be particularly interested in the evaluation of this thing at lambda equals 0.

So what is d En of lambda? d lambda at lambda equals 0 would be psi n at zero, H1, psi n at 0. And therefore, you would say that the En of lambda energies would be the energies at 0, plus lambda, d En of lambda, d lambda at lambda equals 0, plus order lambda squared.

I'm doing just the Taylor expansion of En's of lambda from lambda equals 0. So this thing tells you that En of lambda is equal En of 0, plus-- this derivative you can write it as psi n, lambda, H1, psi n, all at 0. Like that. Plus order lambda squared.

So in this step, I just use the evaluation that we did here. I substituted that and put the lambda in. So that I recognize now that En of lambda is equal to En of 0, plus psi n of 0-- and I can write this as delta H-- psi n of 0, plus order delta H squared.

It's nice to write it this way, because you appreciate more the power of the theorem. The theorem here doesn't assume which value of lambda you have. And you have to have normalized eigenstates.

And you wonder what is it helping you with, if finding the states for every value of lambda is complicated. Well, it certainly helps you to figure out how the energy of the state varies by a simple calculation. Suppose you know the states of the simple Hamiltonian.

Those are the psi's, n, 0. So if you have the psi n 0 over here, you can do the following step. If you want to figure out how it's energy has varied, use this formula in which you compute the expectation value of the change in the Hamiltonian on that state.

And that is the first correction to the energy of the state. So you have this state. You compute the expectation value of the extra piece in the Hamiltonian. And that's the correction to the energy.

It's a little more complicated of course to compute the correction to the state. But that's a subject of perturbation theory. And that's not what we care about right now.

So the reason we're doing this is because actually whatever we're going to have with spin orbit coupling represents an addition to the hydrogen Hamiltonian of a new term. Therefore, you want to know what happens to the energy levels. And the best thing to think about them is to-- if you know the energy levels of this one, well, a formula of this type can let you know what happens to the energy levels after the

perturbation.

There will be an extra complication in that the energy levels that we're going to deal with are going to be degenerate. But let's wait for that complication until it appears. So any questions? Yes?

AUDIENCE: So I would imagine that this would work just as well for time. Because time [INAUDIBLE] a parameter in quantum mechanics. So [INAUDIBLE]

PROFESSOR: Time dependent perturbation theory is a bit more complicated. I'd rather not get into it now. So let's leave it here, in which we don't have time.

And the Schrodinger equation is something like H psi equal [INAUDIBLE] psi, that's all we care. And leave it for that moment. Other questions?

OK. So let's proceed with addition of angular momentum. So first, let me give you the fundamental result of addition of angular momentum.

It's a little abstract, but it's what we really mean by addition of angular momentum. Of angular momentum.

And the main result is the following. Suppose you have a set of operators, J, i, 1, that have the algebra of angular momentum. Of angular momentum. Which is to say Ji1, JJ1, is equal i, h bar, epsilon iJK, JK1.

And this algebra is realized on some state space. On some vector space, V1.

And suppose you have another operator, J-- set of operators actually. Ji2, which have the algebra of angular momentum. I will not write that. On some V2.

OK. Angular momentum, some sets of states. Angular momentum on some other set of states.

Here comes the thing. There is a new angular momentum, which is the sum Ji defined as Ji1, added with Ji2. Now, soon enough you will just write Ji1, plus Ji2.

But let me be a little more careful now. This sum is Ji1, plus 1, tensor Ji2. So i is the

same index.

But here, we're having this operator that we're being defined that we call it the sum. Now how do you sum two operators that act in different spaces? Well, the only thing that you can actually do is sum them in the tensor product.

So the claim is that this is an angular momentum in V1 tensor V2. That is an operator. You see, you have to sum them. So you have to create a space where both can act, and you can sum them.

You cannot sum a thing, an operator that acts on one vector space to an operator that acts on another vector space. You have to create one vector space where both act. And then you can define the sum of the operators.

Sum of operators is a simple thing. So you form the tensor product. In here, this operator gets upgraded in this way, in which in the tensor product it has a 1 for the second input. This one gets upgrade to this way. And this is the sum.

So this is a claim-- this is a definition. And this is a claim. So this has to be proven. So let me prove it.

Ji, JJ. I compute this commutator. So I don't have to do the following. I have to do Ji1, tensor 1, plus 1 tensor Ji2. And then the JJ would be JJ1, tensor 1, plus 1, tensor JJ2.

Have to compute this commutator. Now, an important fact about this result that I'm not trying to generalize, if you had put a minus here, it wouldn't work out. If you would have put a 2 here, it wouldn't work out. If you would have put a 1/2 here, it won't work out.

This is pretty much the only way you can have two angular momenta, and create a third angular momentum. So look at this. It looks like we're going to have to work hard, but that's not true.

Consider this commutator. The commutator of this term with this term. That's 0 actually.

Because if you multiply them in this order, this times that, you get Ji1 times Ji2, because the ones do nothing. You multiply them in the reverse order, you get again, Ji1 times Ji2. This is to say that the operators that originally lived in the different vector spaces commute.

Yes?

AUDIENCE: Since the cross terms between those two are 0-- like you just said, the cross terms are 0. And if you put a minus sign in there, it will cancel. But when you do the multiplications with the second ones, why can't you put a minus sign in there? [INAUDIBLE]

PROFESSOR: In the whole thing? In this definition, a minus sign?

AUDIENCE: Yeah.

PROFESSOR: Well, here if I put a minus-- it's like I'm going to prove that this works. So if-- I'm going to get an angular momentum. If I put a minus sign to angular momentum, I ruin the algebra here.

I put a minus minus, it cancels. But then I get a minus sign here. So I cannot really even change a sign.

So any way, these are operators acting on different spaces. They commute. It's clear they commute. You just multiply them, and see that.

These one's commute as well. The only ones that don't commute are this with this. And that with that. So let me just write them.

Ji1, tensor 1, with JJ1, tensor 1. Plus this one, 1 tensor Ji2, 1 tensor JJ2.

OK, next step is to realize that actually the 1 is a spectator here. Therefore, this commutator is nothing but the commutator Ji1 with JJ1, tensor 1. You can do it.

If you prefer to write it, write it. This product is Ji times JJ, tensor 1. And the other product is JJ, Ji, tensor 1. So the tensor 1 factors out.

Here the tensor 1 also factors out. And you get an honest commutator, Ji2, JJ2. So one last step.

This is i, h bar, epsilon, iJK. I'll put a big parentheses. JK1, tensor 1, for the first one. Because J1 forms an angular momentum algebra.

And here, 1 tensor JK2. And this thing is i, h bar, epsilon, iJK. The total angular momentum, K. And you've shown the algebra works out.

Now most people after a little practice, they just say, oh, Ji is J1 plus J2, J1 plus J2. J1 and J2 don't commute.

J2 and J1-- I'm sorry. J1 and J2 commute. J2 and J1 commute. Therefore you get this 2, like that.

And this gives you-- J1 and J1 gives you J1. J2 and J2 gives you J2, so the sum works out. So most people after a little practice just don't put all these tensor things. But at the beginning it's nice to just make sure that you understand what these tensor things do.

All right. So that's our main theorem-- that you start with one angular momentum on a state space. Another angular momentum that has nothing to do perhaps with the first on another vector space. And on the tensor product you have another angular momentum, which is the sum.

All right. So now, we do spin orbit coupling to try to apply these ideas. So for spin orbit coupling, we will consider the hydrogen atom coupling. And the new term in the Hamiltonian, mu dot B.

The kind of term that we've done so much in this semester. We've looked over magnetic ones. So which magnetic moment at which B? There was no B in the hydrogen atom.

Well, there's no B to begin with. But here is one where you can think there is a B. First, this will be the electron dipole moment. Magnetic dipole moment.

So we have a formula for it. The formula for it is the mu of the electron is minus E over m, times the spin of the electron. And I actually will use a little different formula that is valued in Gaussian units. ge over mC, S, in Gaussian units.

And g is the g factor of the electron, which is 2. I'm sorry. There's a 2 here.

OK. So look what I've written. I don't want to distract you with this too much. But you know that the magnetic dipole of the electron is given by this quantity.

Now, you could put a 2 up, and a 2 down. And that's why people actually classically there seems to be a 2 down. But there's a 2 up, because it's an effect of the electron. And you have this formula.

The only thing I've added in that formula is a factor of C that is because of Gaussian units. And it allows you to estimate terms a little more easily. So that's the mu of the electron.

But the electron apparently would feel no magnetic field. You didn't put an external magnetic field. Except that here you go in this way of thinking-- you think suppose you are the electron. You see a proton, which is a nucleus going around you. And a proton going around you is a current going around you. It generates a magnetic field. And therefore, you see a magnetic field created by the proton going around you. So there is a magnetic field. And there's a magnetic field experienced by the electron-- felt by electron.

So you can think of this, the electron. Here is the proton with the plus charge, and here's the electron. And the electron is going around the proton. Now, from the viewpoint of the electron, the proton is going around him.

So here is the proton. Here is the electron going like that. From the viewpoint of the electron, the proton is going like this. Also, from the viewpoint of the electron, the proton would be going in this direction and creating a magnetic field up.

And the magnetic field up corresponds actually to the idea that the angular

momentum of the electron is also up-- L of the electron is also up. So the whole point of this thing is that somehow this magnetic field is proportional to the angular momentum. And then, L will come here. And here, you have S. So you have L dot S. That's the fine structure coupling.

Now let me do a little of this so that we just get a bit more feeling, although it's unfortunately a somewhat frustrating exercise. So let me tell you what's going on. So consider the electron. At some point, look at it and draw a plane.

So the electron-- let's assume it's going down. Here is the proton. It's going around in circles. So here, it's going down. The electron is going down. Electron, its velocity of the electron is going down.

The proton is over here. And the electron is going around like that. The proton would produce an electric field of this form. Now, in relativity, the electric and magnetic fields seen by different observers are different. So there is this electric field that we see. We sit here, and we see in our rest frame this proton creates an electric field.

And then, from the viewpoint of the electron, the electron is moving. And there is an electric field. But whenever you are moving inside an electric field, you also see a magnetic field generated by the motion, by relativistic effects.

The magnetic field that you see is roughly given to first order in relativity by V cross E over c. So V cross E, VE V cross E over c up-- change sign because of this. And the magnetic field consistently, as we would expect, goes in this direction.

So it's consistent with the picture that we developed that if you were the electron, the proton, would be going around in circles like that and the magnetic field would be up. Now here I can change the sign by doing E cross V over c. So this is the magnetic field seen by the electron.

OK, so we need a little more work on that magnetic field by calculating the electric field. Now, what is the electric field? Well, the scalar potential for the hydrogen atom, we write it as minus e squared over r. It's actually not quite the scalar

potential. But it is the potential energy. It has one factor of e more than what the scalar potential is.

Remember, the scalar potential in electromagnetism is charge divided by r. So it has one factor of e more. What is the derivative of this potential? With respect to r, it's e squared over r squared. So the electric field goes like e over r squared.

So the electric field is equal to dV dr divided by e. That's the magnitude of the electric field. And its direction is radial from the viewpoint of the proton. The electric field is here.

So this can be written as r vector divided by r. Therefore, the magnetic field will-- [INAUDIBLE] this. The magnetic field now can be calculated. And we'll see what we claimed was the relation with angular momentum. Because B prime is now E cross V. So you have 1 over ec 1 over r dV dr.

I've taken care of this. And now I just have r cross V. Well, r cross V is your angular momentum if you had p here. So we borrow a factor of the mass of the electron, ecm 1 over r dV dr L, L of the electron. p equals mv.

So we have a nice formula for B. And then, we can go and calculate delta H. Delta H would then be minus mu dot B. And that would be ge over 2mc spin dot L-- mu was given here-- S dot L 1 over r dV dr. And that is the split spin orbit interaction.

Now, the downside of this derivation is that it has a relativistic error. There's a phenomenon called Thomas precession that affects this result. We didn't waste our time. The true result is that you must subtract from this g 1. So g must really be replaced by g minus 1. Since g is approximately 2 for the electron, the true result is really 1/2 of this thing. So this should not be in parentheses, but true result is this.

And the mistake that is done in calculating this spin orbit coupling is that this spin orbit coupling affects precession rates. All these interactions of magnetic dipoles with magnetic fields affect precession rates. And you have to be a little more careful here that the system where you've worked, the electron rest frame is not quite an inertial system. Because it's doing circular motion. So there's an extra correction

that has to be done. Thomas precession or Thomas correction it's called.

And it would be a detour of about one hour in special relativity to do it right. So Griffiths doesn't do it. I don't think Shankar does it. Pretty much graduate books do it. So we will not try to do better. I mentioned that fact that this really should be reduced to one half of its value. And it's an interesting system to analyze.

So Thomas precession is that relativistic correction to precession rates when the object that is precessing is in an accelerated frame. And any rotating frame is accelerated. So this result needs correction.

OK, but let's take this result as it is-- instead of g, g minus 1. Let's not worry too much about it. And let's just estimate how big this effect is. It's the last thing I want to do as a way of motivating this subject. So delta H is this. Let's estimate it.

Now for estimates, a couple of things are useful to remember, that Bohr radius is h squared over me squared. We did that last time. And there's this constant that is very useful, the fine structure constant, which is e squared over hc. And it's about 1 over 137. And it helps you estimate all kinds of things.

Because it's a rather complicated number to evaluate, you need all kinds of units and things like that. So the charge of the electron divided by hc being 1 over 137 is quite nice. So let's estimate delta H. Well, g we won't worry-- 2, 1, doesn't matter. e mc-- so far, that is kind of simple.

Then we have S dot L. Well, how do I estimate S dot L? I don't do too much. S spin is multiples of h bar. L for an atomic state will be 1, 2, 3, so multiples of h bar. So h bar squared, that's it for S dot L.

1 over r is 1 over r. dV dr is e squared over r squared. And that's it. But here, instead of r, I should put the typical length of the hydrogen atom, which is a0. So what do I get? I'm sorry, I made a mistake here.

AUDIENCE: Yeah, it's up there.

PROFESSOR: Oh, I made a mistake here in that I didn't put this factor, 1 over ecm. So the e cancels. And this is the result here-- g over 2m squared c squared S dot L 1 over r dV dr. So let me start again.

1 over m squared c squared h bar squared 1 over r dV dr-- that much I got right. So this is roughly 1 over [INAUDIBLE] of the electron squared c squared e squared over a0 cubed h squared-- still quite messy, but not that terrible. So in order to get an idea of how big this is, the ground state energy of the hydrogen atom was e squared over 2a0. So let's divide delta H over the ground state energy. And that's how much?

Well, we have all this quantity, 1 over m squared c squared e squared a0 cubed h squared. And now, we must divide by e squared over a0 like this. Well, the e squareds cancel. And we get h squared over m squared c squared a0 squared.

You need to know what a0 is. Let's just boil it down to the simplest thing, so h squared m squared c squared. a0 squared would be h to the fourth m squared e to the fourth. So this is actually e to the fourth over h squared c squared, or e squared over hc squared, which is alpha squared. Whew-- lots of work to get something very nice.

The ratio of the spin orbit coupling to the ground state energy is 1 over alpha squared. It's alpha squared, which is 1 over 137 squared. So it's a pretty small thing. It's about 1 over 19,000. So when this is called fine structure of the hydrogen atom, it means that it's in the level in your page that you use a few inches to plot the 13.6 electron volts-- well, you're talking about 20,000 times smaller, something that you don't see. But of course, it's a pretty important thing.

So all in all, in the conventions of-- this is done in Gaussian units. In SI units, which is what Griffiths uses, delta H is e squared over 8 pi epsilon 0 1 over m squared c squared r cubed S dot L. That's for reference.

This is Griffiths. But this is correct as well. This is the correct value. This is the correct value already taking into account the relativistic correction.

So here, you're supposed to let g go to g minus 1. So you can put the 1 there, and it's pretty accurate. All right, so what is the physics question we want to answer with this spin orbit coupling?

So here it comes. You have the hydrogen atom spectrum. And that spectrum you know. At L equals 0, you have one state here. Then, that's n equals 1, n equals 2. You have one state here and one state here at L equals 1. Then n equals 3, they start getting very close together. n equals 4 is like that.

Let's consider if you want to have spin orbit coupling, we must have angular momentum. And that's L. And therefore, let's consider this state here. l equals 1, n equals 1-- n equals 2, I'm sorry. What happens to those states, is the question.

First, how many states do you have there and how should you think of them? Well actually, we know that an l equals 1 corresponds to three states. So you'd have lm with l equals 1. And then m can be 1, 0, or minus 1. So you have three states.

But there's not really three states. Because the electron can have spin. So here it is, a tensor product that appears in your face because there is more than angular momentum to the electron. There's spin. And it's a totally different vector space, the same particle but another vector space, the spin space.

So here, you have the possible spins of the electron. So that's another angular momentum. And well, you could have the plus/minus states, for example. So you have three states here and two states here. So this is really six states, so six states whose fate we would like to understand due to this spin orbit coupling.

So to use the language of angular momentum, instead of writing plus/minus, you could write Smz, if you will-- ms I will call, spin of s. You have here spin of 1/2 and states 1/2 or minus 1/2. This is the up. When the z component of the spin that we always call m-- m now corresponds to the z component of angular momentum. So in general, even for spin, we use m. And we have that our two spin states of the electron are spin 1/2 particle with plus spin in the z direction, spin 1/2 particle with minus spin in the z direction.

We usually never put this 1/2 here. But now you have here really three states-- 1, 1, 1, 0, 1, minus 1, the first telling you about the total angular momentum. Here, the total spin is 1/2. But it happens to be either up or down. Here, the total angular momentum is 1. But it happens to be plus 1, 0, or minus 1 here.

So these are our six states. You can combine this with this, this with that, this with this, this with that. You make all the products. And these are the six states of the hydrogen atom at this level. And we wish to know what happens to them.

Now, this correction is small. So it fits our understanding of the perturbation theory of Feynman-Hellman in which we try to find the corrections to these things. Our difficulty now is a little serious, however. It's the fact that Feynman-Hellman assumed that you had a state. And it was an eigenstate of the corrected Hamiltonian as you moved along. And then, you could compute how its energy changes.

Here, unfortunately, we have a much more difficult situation. These six states that I'm not listing yet, but I will list very soon, are not obviously eigenstates of delta H. In fact, they are not eigenstates of delta H. They're degenerate states, six degenerate states, that are not eigenstates of delta H. Therefore, I cannot use the Feynman-Hellman theorem until I find what are the combinations that are eigenstates of this perturbation.

So we are a little bit in trouble. Because we have a perturbation for which these product states-- we call them uncoupled bases-- are not eigenstates. Now, we've written this operator a little naively. What does this operator really mean, S dot L?

In our tensor products, it means S1 tensor L1. Actually, I'll use L dot S. I'll always put the L information first and the S information afterward. So L dot S is clearly an operator that must be thought to act on the tensor product. Because both have to act. S has to act and L has to act. So it only lives in the tensor product. So what does it mean?

It means this-- S2 L2 plus S3 L3, or sum over i Si tensor Li. So this is the kind of

thing that you need to understand-- how do you find for this operator's eigenstates here? So that is our difficulty. And that's what we have to solve.

We're going to solve it in the next half hour. So it's a complicated operator, L dot S. But on the other hand, we have to use our ideas that we've learned already about summing angular momenta. What if I define J to be L plus S, which really means L tensor 1 plus 1 tensor S?

So this is what I really mean by this operator. J, as we've demonstrated, will be an angular momentum, because this satisfies the algebra of angular momentum and this satisfies the algebra of angular momentum. So this thing satisfies the algebra of angular momentum.

And why do we look at that term? Because of the following reason. We can square it-- JiJi. Now we would have to square this thing. How do you square this thing? Well, there's two ways. Naively-- L squared plus L squared plus 2L dot S-- basically correct.

But you can do it a little more slowly. If you square this term, you get L squared tensor 1. If you square this term, you get 1 tensor S squared. But when you do the mixed products, you just must take the i's here and the i's here and multiply them. So actually, you do get two i's, the sum over i Li tensor Si.

This is sum over i. This is J squared. So basically, what I'm saying is that J squared naively is L squared plus S squared plus our interaction 2L dot S defined property. So L dot S is equal to 1/2 of J squared minus L squared minus S squared. And that tells you all kinds of interesting things about L dot S.

Basically, we can trade L dot S for J squared, L squared, and S squared. L squared is very simple, and S squared is extremely simple as well. Remember, L squared commutes with any Li. So L squared with any Li is equal to 0. S squared with any Si is equal to 0.

And Li's and Si's commute. They live in different worlds. So L squared and Si's commute. S squareds and Li's commute. These things are pretty nice and simple.

So let's think now of our Hamiltonian and what is happening to it. Whenever we had the hydrogen atom, we had a set of commuting observables H, L squared, and Lz. It's a complete set of commuting observables. Now, in the hydrogen atom, you could add to it S squared and Sz.

We didn't talk about spin at the beginning, because we just considered a particle going around the hydrogen atom. But if you have spin, the hydrogen atom Hamiltonian, the original one, doesn't involve spin in any way. So certainly, Hamiltonian commutes with spin, with spin z. L and S don't talk, so this is the complete set of commuting observables.

But what happens to this list? This is our problem for H0, the hydrogen atom, plus delta H that has the S dot L. Well, what are complete set of commuting observables? This is a very important question. Because this is what tells you how you're going to try to organize the spectrum. So we could have H, the total, H total. And what else?

Well, can I still have L squared here? Can I include L squared and say it commutes with the total H? A little worrisome, but actually, you know that L squared commutes with the original Hamiltonian. Now, the question is whether L squared commutes with this extra piece. Well, but L squared commutes with any Li. And it doesn't even talk to S. So L squared is safe. L squared we can keep.

OK, S squared-- can we keep S squared? Well, S squared was here. So it commuted with the Hamiltonian, and that was good. S squared commutes with any Si, and it doesn't talk to L. So S squared can stay.

But that's not good enough. We won't be able to solve the problem with this still. We need more. How about Lz? It was here, so let's try our luck. Any opinions on Lz-- can we keep it or not? Yes.

AUDIENCE: I don't think so. Because in the J term, we have Lx's and Ly's, which don't commute with Lz.

PROFESSOR: Right, it can't be kept. Here, this term has SxLx plus SyLy plus SzLz. And Lz doesn't commute with this one. So no, you can't keep Lz-- no good. On the other hand, let's think about J squared. J squared is here. And J squared commutes with L squared and with S squared.

J squared, therefore, is-- well, let me say it this way. Here is L dot S, which is our extra interaction. Here we have this thing. I would like to say on behalf of J squared that we can include it here, J squared, because J squared is really pretty much the same as L dot S up to this L squared and S squared.

But J squared commutes with L squared and S squared. I should probably write it there. J squared commutes with L squared. And J squared communicates with S squared that we have here. And moreover, we have over here that J squared therefore will commute, or it's pretty much the same, as L dot S. J squared with L dot S would be J squared times this thing, which is 0.

So J squared commutes with this term. And it commutes with the Hamiltonian, your original hydrogen Hamiltonian. So J squared can be added here. J square is a good operator to have. And now we can get one more kind of free from here. It's Jz. Z

Because Jz commutes with J squared. Jz commutes with these things. And Jz, which is a symmetry of the original Hamiltonian, also commutes with our new interaction, the L dot S, which is proportional to J squared.

So you have to go through this yourselves probably even a little more slowly than I've gone. Just check that everything that I'm saying about whatever commutes commutes. So for example, when I say that J squared commutes with L dot S, it's because I can put instead of L dot S all of this. And go slowly through this.

So this is actually the complete set of committing observables. And it's basically saying to us, try to diagonalize this thing with total angular momentum. So it's about time to really do it. We haven't done it yet. But now the part that we have to do now, it's kind of a nice exercise. And it's fun.

Now, there's one problem in the homework set that sort of uses this kind of thing.

And I will suggest there to Will and Aram that tomorrow, they spend some time discussing it and helping you with it. The last problem in the homework set would've been better if you had a little more time for it and you had more time to digest what I'm doing today. But nevertheless, go to recitation, learn more about the problem. It will not be all that difficult.

OK, so we're trying now to finally form another basis of states. We had these six states. And we're going to try to organize them in a better way-- as eigenstates of the total angular momentum L plus S. So I'm going to write them here in this way. Here is one of the states of this L equals 1 electron, the 1, 1 coupled to the 1/2, 1/2.

Here are two more states- 1, 0, 1/2, 1/2, 1, 1, 1/2, minus 1/2, so the 1, 0 with the top, the 1, 1 with the bottom. Here are two more states-- 1, 0 with 1/2, minus 1/2 and 1, minus 1 with 1/2, 1/2. And here is the last state-- 1, minus 1 with 1/2, minus 1.

These are our six states. And I've organized them in a nice way actually. I've organized them in such a way that you can read what is the value of Jz over h bar. Remember, Jz is 1 over h bar Lz plus Sz. So what is it? These are, I claim, eigenstates of Jz. Why? Because let's act on them. Suppose I act with Jz on this state. The Lz comes here and says, 1. The Sz comes here and says, 1/2.

So the sum of them give you Jz over h bar equal to 3/2. And that's why I organized these states in such a way that these second things add up to the same value-- 0 and 1/2, 1 and minus 1/2. So if you act with Jz on this state, it's an eigenstate with Jz. Here, 0 contribution, here 1/2. So this is with plus 1/2.

Here, you have 0 and minus 1/2, minus 1, and that is minus 1/2. And here you have minus 3/2. OK, questions. We've written the states. And I'm evaluating the total z component of angular momentum. And these two states are like that.

So what does our theorem guarantee for us? Our theorem guarantees that we have-- in this tensor product, there is an algebra of angular momentum of the Jz operators. And the states have to fall into representations of those operators. So

you must have angular momentum multiplets.

So at this moment, you can figure out what angular momentum you're going to get for the result. Here we obtained a maximum Jz of 3/2. So we must get a J equals 3/2 multiplet. Because a J equaling 3/2 multiplets has Jz 3/2, 1/2, minus 1/2, and 0. So actually, this state must be the top state of the multiplet. This state must be the bottom state of the multiplet.

I don't know which one is the middle state of the multiplet and which one is here. But we have four states here, four states. So one linear combination of these two states must be, then, that Jz equals 1/2 state of the multiplet. And one inner combination of these two states must be that Jz equals minus 1/2 state of the multiplet.

Which one is it? I don't know. But we can figure it out. We'll figure it out in a second. Once you get this J 3/2 multiplet, there will be one linear combination here left over and one linear combination here left over. Those are two state, one with Jz plus 1/2 and one with Jz equals minus 1/2. So you also get a J equals 1/2 multiplet.

So the whole tensor product of six states-- it was the tensor product of a spin 1 with a spin 1/2. So we write it like this. The tensor product of a spin 1 with a spin 1/2 will give you a total spin 3/2 plus total spin 1/2-- funny formula. Here is the tensor product, the tensor product of these three states with these two states. This can be written as 3 times 2 is equal to 4 plus 2 in terms of number of states. The tensor product of this spin 1 and spin 1/2 gives you a spin 3/2 multiplet with four states and a spin 1/2 multiplet with two states.

So how do you calculate what are the states themselves? So the states themselves are the following. All right, here I have them. I claim that the J equals 3/2 states, m equals 3/2 states, the top state of that multiplet can only be the state here, the 1, 1 tensor 1/2, 1/2. And there's no way any other state can be put on the right. Because there's no other state with total z component of angular momentum equals 3/2. So that must be the state.

Similarly, the J equals 3/2, m equals minus 3/2 state must be the bottom one-- 1,

minus 1, 1/2, minus 1/2. The one that we wish to figure out is the next state here, which is the J equals 3/2, m equals 1/2. It's a linear combination of these two. But which one? That is kind of the last thing we want to do. Because it will pretty much solve the rest of the problem.

So how do we solve for this? Well, we had this basic relation that we know how to lower or raise states of angular momentum-- m times m plus/minus 1 J-- I should have written it J plus/minus Jm equals h bar square root. More space for everybody to see this-- J times J plus 1 minus m times m plus/minus 1. Close the square root-- Jm plus/minus 1. So what I should try to do is lower this state, try to find this state by acting with J minus.

So let me try to lower the state, so J minus on this state, on J equals 3/2, m equals 3/2. I can go to that formula and write it as h bar square root. J is 3/2, so 3/2 times 5/2 minus m, which is 3/2, times m minus 1, 1/2. We're doing the minus-- times the state 3/2, 1/2. So the state we want is here. And it's obtained by doing J minus on that.

But we want the number here. So that's why I did all these square roots. And that just gives h bar square root of 3, 3/2, 1/2. Well, that still doesn't calculate it for me. But it comes very close. So you have it there. Now I want to do this but using the right hand side.

So look at the right hand side. We want to do J minus, but on 1, 1 tensor 1/2, 1/2. So I applied J minus to the left hand side. Now we have to apply J minus to the right hand side. But J minus is L minus plus S minus on 1, 1 tensor 1/2, 1/2.

When this acts, it acts on the first. So you get L minus on 1, 1 tensor 1/2, 1/2. And in the second term, you get plus 1, 1 tensor S minus on 1/2, 1/2. Now, what is L minus on 1, 1? You can use the same formula. It's 1, 1. And it's an angular momentum.

So it just goes on and gives you h bar square root of 1 times 2 minus 1 times 0. 1, 0-- it lowers it-- times 1/2, 1/2. Let me go here-- plus 1, 1. And what is S minus on this? Use the formula with J equals 1/2. So this is h bar square root of 1/2 times 3/2

minus 1/2 times minus 1/2 times 1/2 minus 1/2.

Whew-- well not too difficult. But this gives you h over square root of 2, 1, 0 tensor 1/2, 1/2 plus just h bar. This whole thing is 1-- 1, 1 tensor 1/2, minus 1/2. OK, stop a second to see what's happened.

We had this equality. And we acted with J minus. Acting on the left, it gives us a number times the state we want. Acting on the right, we got this. So actually, equating this to that, or left hand side to right hand side, we finally found the state 3/2, 1/2.

So the state 3/2, 1/2 is as follows. 3/2, 1/2 is-- you must divide by that square root. So you get the square root of 3 down. The h bars cancel. So here it is, a very nice little formula-- 2 over 3, 1, 0 tensor 1/2, 1/2 plus 1 over square root of 3, 1, 1 tensor 1/2, minus 1/2.

So we have the top state of the multiplet. We have the next state of the multiplet. We have-- I'm sorry, the top state of the multiplet was this. You have the bottom state of the multiplet, the middle state of the multiplet. What you're missing is the bottom and the middle term.

And this one can be obtained in many ways. One way would be to raise this state. The minus 3/2 could be raised by one unit and do exactly the same thing. Well, the result is square root of 2 over 3, 1, 0 tensor 1/2, minus 1/2 plus 1 over square root of 3. That square root of 2 doesn't look right to me now. I must have copied it wrong. It's 1 over square root of 3-- 1 over square root of 3, 1, minus 1 tensor 1/2, 1/2.

So you've built that whole multiplet. And this state, as we said, was a linear combination of the two possible states. This 3 minus 1/2 was a linear combination of these two possible states.

So the other states that are left over, the other linear combinations, form the J equals 1/2 multiplet. So basically, every state must be orthogonal to each other. So the other state, the 1/2, 1/2 and the 1/2, minus 1/2 of the J equals 1/2 multiplet must

be this orthogonal to this. And this must be orthogonal to that.

So those formulas are easily found by orthogonality. So I'll conclude by writing them-- minus 1 over square root of 3, 1, 0, 1/2, 1/2 plus the square root of 2 over 3, 1, 1, 1/2, minus 1/2. And here, you get 1 over square root of 3, 1, 0, 1/2, minus 1/2 minus 2 over square root of 3, 1, minus 1 tensor 1/2, 1/2.

So lots of terms, a little hard to read-- I apologize. Now, the punchline here is that you've found these states. And the claim is that these are states in which L dot S is diagonal. And it's kind of obvious that that should be the case. Because what was L dot S?

So one last formula-- L dot S equals 1/2 of J squared minus L squared minus S squared. Now, in terms of eigenvalues, this is 1/2 h squared J times J plus 1 minus L times L plus 1 minus S times S plus 1. Now, all the states that we built have definite values of J squared, definite values of S squared. Because L was 1. And S is 1/2.

So here you go h squared over 2 J times J plus 1 minus 1 times 2 is 2 minus 1/2 times 3/2 is 3/4. And that's the whole story. The whole story in a sense has been summarized by this. We have four states with J equals 3/2 and two states with J equals 1/2. So these six states that you have here-- split because of this interaction into four states that have J equal to 3/2 and two states that have J equal to 1/2.

And you plug the numbers here. And that gives you the amount of splitting. So actually, this height that this goes up here is h squared over 2. And this is minus h squared by the time you put the numbers J, 3/2, and 1/2.

So all our work was because the Hamiltonian at the end was simple in J squared. And therefore, we needed J multiplets. J multiplets are the addition of angular momentum multiplets.

In a sense, we don't have to construct these things if you don't want to calculate very explicit details. Once you have that, you have everything. This product of angular momentum 1, angular momentum 1/2 gave you total angular momentum

3/2 and 1/2-- four states, two states. So four states split one way, two states split the other way, and that's the end of the story. So more of this in recitation and more of this all of next week. We'll see you then.